

Existence, uniqueness, and stability of optimal portfolios of eligible assets

MICHEL BAES¹

Department of Mathematics, ETH Zurich, Switzerland

PABLO KOCH-MEDINA², COSIMO MUNARI³

Center for Finance and Insurance, University of Zurich, Switzerland

February 8, 2017

Abstract

We study the existence of portfolios of traded assets making a given financial institution pass some pre-specified (internal or external) regulatory test. In particular, we are interested in the existence of optimal portfolios, i.e. portfolios that allow to pass the test at the lowest cost, and in their sensitivity to changes in the underlying capital position. This naturally leads to investigate the continuity properties of the set-valued map associating to each capital position the corresponding set of optimal portfolios. We pay special attention to inner semicontinuity, which is the key continuity property from a financial perspective. This property is always satisfied if the test is based on a polyhedral risk measure such as Expected Shortfall, but it generally fails, even in a convex world, if we depart from polyhedrality. In this case, the optimal portfolio map may even fail to admit a continuous selection. Our results have applications to capital adequacy, pricing and hedging, and capital allocation. In particular, we allow for regulatory tests designed to capture systemic risk.

Keywords: risk measures, portfolio selection, capital adequacy, systemic risk, pricing, hedging, capital allocation.

Mathematics Subject Classification: 91B30, 91B32

1 Introduction

This paper investigates the properties of a type of set-valued maps that arises in several important financial problems, including capital adequacy, pricing and hedging, and systemic risk regulation. For the sake of definiteness we will motivate our topic and state our results in the language of capital adequacy. The link to the other applications is established at the end of Section 2.

A financial institution is required by regulators to hold an adequate capital buffer to protect liability holders from the risk of default. Whether the capital base of an institution is adequate or not is established by a regulatory capital adequacy test that is usually based on VaR (Value-at-Risk) or ES (Expected Shortfall). Associated to this test is a risk measure, or capital requirement rule, that determines the minimum amount of capital an institution has to *raise* to pass the regulatory test. This minimum amount

¹Email: mbaes@math.ethz.ch

²Email: pablo.koch@bf.uzh.ch

³Email: cosimo.munari@bf.uzh.ch

will, however, depend on how this capital is *invested* once raised. Hence, the risk measure makes sense as a rule to determine capital requirements only once the allocation procedure has been specified. In line with the original framework described in the seminal paper by Artzner, Delbaen, Eber, Heath (1999), the bulk of the literature on risk measures assumes, either explicitly or implicitly, that raised capital is held either in cash or invested in a single pre-specified traded asset, which, following Artzner, Delbaen, Koch-Medina (2009), we call the *eligible asset*. It is worth emphasizing that raising the amount of capital determined by the risk measure only ensures acceptability of the institution if this amount is actually invested in the pre-specified eligible asset.

As it has been pointed out in Artzner, Delbaen, Koch-Medina (2009) and, more recently, in Farkas, Koch-Medina, Munari (2015), limiting the investment choice to a *single* eligible asset instead of allowing investments in *portfolios* of multiple eligible assets is inefficient in that it generally leads to higher capital requirements. Note that a lower capital requirement does not mean that the institution is less safe. Indeed, the institution continues to satisfy the same capital adequacy standard. It is just that to reach acceptability less capital is needed if one has more choices on how to invest it.

In the context of multiple eligible assets, we will address the following three theoretical questions that are critical from an operational perspective:

- **Existence of optimal portfolios.** Do optimal portfolios of eligible assets, i.e. portfolios that allow to pass the regulatory test at the minimum cost, exist at all?
- **Uniqueness of optimal portfolios.** In case optimal portfolios exist, are they unique or does management have several alternatives from which to choose?
- **Stability of optimal portfolios.** In case several optimal portfolios exist, how robust is the choice of a specific portfolio? In other words, if we base our allocation on a slightly misestimated capital position, how confident can we be that this choice will not be “too wrong” for the actual position?

While properties of risk measures themselves have been subjected to detailed scrutiny, it seems that the above questions have not been addressed before. However, in our view, any analysis of capital requirements would not be complete without having answered them. From a theoretical perspective, failing to answer these questions would amount to studying an optimization problem focusing only on the optimal value of the objective function and not paying attention to the structure of the solution set. From a practical perspective, finding an answer to these questions is critical since, for a capital regime to be operationally effective, it is necessary to make sure that managers know which actions they can undertake to meet capital requirements and that these actions are robust with respect to misestimations.

The main objective of this paper is to provide answers to the above questions. The first two questions are addressed in Section 4, where we provide a variety of necessary and sufficient conditions for both existence and uniqueness of optimal portfolios. In particular, we show that uniqueness is always guaranteed if the capital adequacy test is based on a strictly-convex risk measure and provide a dual characterization of uniqueness in the case of a polyhedral risk measure such as ES. The last question is addressed in Section 5, where we establish a link between the above intuitive notion of “stability” or “robustness”, which stipulates that a small perturbation of the capital position should not yield a significant change in the optimal portfolio structure, and a variety of (semi)continuity properties of the set-valued map that assigns to each capital position the corresponding set of optimal portfolios. We focus on three continuity properties, namely outer semicontinuity, upper semicontinuity, and inner or lower semicontinuity. We first show that outer semicontinuity is always satisfied and then provide necessary and sufficient conditions for upper semicontinuity. Finally, we turn to inner semicontinuity, which constitutes the key stability property in our financial context. We show that inner semicontinuity may fail if the capital adequacy test is based on a nonconvex risk measure, such as VaR, as well as a convex risk measure. However, if the risk measure is either polyhedral, such as ES, or strictly convex, one can always ensure inner semicontinuity.

2 The modelling framework

In this section we present the mathematical model which will be used to formalize the main questions addressed in this paper. In the first part we focus on solvency regulation and capital adequacy, which constitutes our motivating framework. In a second part we establish a bridge between our problem and other problems arising in finance, which allows to interpret our results from a broader financial perspective.

2.1 Solvency regulation and capital adequacy

We consider a one-period economy with initial date $t = 0$ and terminal date $t = 1$. The uncertainty about the future state of the economy is modelled by a finite state space

$$\Omega = \{\omega_1, \dots, \omega_N\}.$$

Each element of Ω describes a possible state of the economy at time 1.

Capital positions of financial institutions

At time 0, financial institutions are assumed to issue liabilities promising a state-contingent payoff and invest in assets that entitle to receiving a state-contingent payoff. At time 1, they receive the payoffs of the assets they hold and use them to cover the payoffs of their liabilities. Any excess of assets above liabilities flows to the owners as profit. However, liabilities are honoured only in those states of the economy where the payoffs of the assets suffice to do so, i.e. the owners have limited liability. Since they are state contingent, the payoff of *assets* at time 1 can be described by a random variable of the form $A : \Omega \rightarrow \mathbb{R}_+$ and the payoff of *liabilities* by a random variable $L : \Omega \rightarrow \mathbb{R}_+$. The *capital position* of the financial institution at time 1 corresponds to assets net of liabilities and is described by the random variable

$$X = A - L.$$

We will denote by \mathcal{X} the vector space of random variables defined on Ω and we will interpret its elements either as capital positions, or payoffs of assets, or as payoffs of liabilities. Note that under these interpretations, a positive value of a random variable represents a payment to be received and a negative value an obligation to be met. The space \mathcal{X} becomes a partially-ordered normed space when equipped with the canonical pointwise order and with the maximum norm. For clarity we provide in the appendix a brief overview of our notation and of selected notions and facts from convex analysis we will freely use throughout the paper.

Acceptance sets

Regulators divide financial institutions into two groups, namely those that are deemed adequately capitalized and those that are not. This amounts to specifying an *acceptance set* or *capital adequacy test*, i.e. a set $\mathcal{A} \subset \mathcal{X}$ of capital positions that are deemed acceptable from a regulatory perspective. A financial institution is then adequately capitalized if its capital position belongs to the set \mathcal{A} . An acceptance set is assumed to be a nonempty, proper subset of \mathcal{X} satisfying the *monotonicity property*

$$(A1) \quad X \in \mathcal{A}, Y \geq X \implies Y \in \mathcal{A}.$$

Once a capital position X has been accepted, any position that dominates X in every state of the economy must also be accepted. We will also assume that

$$(A2) \quad \mathcal{A} \text{ is closed and } 0 \in \mathcal{A}.$$

Asking that 0 is acceptable amounts to viewing an institution acceptable if in all states of the economy the payoffs of its assets are identical to the payoffs required to meet liabilities. The above properties are widely recognized as the minimal properties an acceptance set should satisfy. Acceptance sets that are convex are important because they capture diversification effects. Within the class of convex acceptance sets, polyhedral acceptance sets are particularly tractable. Following the terminology introduced by Artzner, Delbaen, Eber, Heath (1999), convex acceptance sets that are also cones are said to be *coherent*.

The most prominent acceptance sets in a regulatory environment are those based on VaR (Value-at-Risk) and those based on ES (Expected Shortfall). VaR is the risk metric historically adopted by the Basel Committee on Banking Supervision — the reference regulatory regime for the banking sector — and by the Solvency II framework — the regulatory regime for insurance companies within the EU. ES is the basis for the Swiss Solvency Test — the regulatory framework for insurance companies in Switzerland — and is set to be the risk metric for market risk in the third Basel Accord.

Value-at-Risk. Assume that we assign to each state of the economy a probability of occurrence so that a probability measure \mathbb{P} is defined on the power set of Ω . The *Value-at-Risk* of a position $X \in \mathcal{X}$ at the level $\alpha \in (0, 1)$ is defined as usual by

$$\text{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}; \mathbb{P}(X + m < 0) \leq \alpha\}.$$

Hence, $\text{VaR}_\alpha(X)$ is nothing but the negative of the upper α -quantile of X . The corresponding acceptance set is the closed cone given by

$$\mathcal{A}_{\text{VaR}}(\alpha) := \{X \in \mathcal{X}; \text{VaR}_\alpha(X) \leq 0\} = \{X \in \mathcal{X}; \mathbb{P}(X < 0) \leq \alpha\}.$$

In particular, acceptability boils down to checking whether the probability of default or loss of a certain position does not exceed the threshold α . Note that, in general, $\mathcal{A}_{\text{VaR}}(\alpha)$ is not convex. \square

Expected Shortfall. Let \mathbb{P} be as in the preceding example. The *Expected Shortfall* of $X \in \mathcal{X}$ at the level $\alpha \in (0, 1)$ is defined by

$$\text{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) d\beta.$$

The corresponding acceptance set is the closed polyhedral cone given by

$$\mathcal{A}_{\text{ES}}(\alpha) := \{X \in \mathcal{X}; \text{ES}_\alpha(X) \leq 0\} = \bigcap_{\mathbb{Q} \in \mathcal{Q}_\alpha} \{X \in \mathcal{X}; \mathbb{E}_{\mathbb{Q}}[X] \geq 0\},$$

where \mathcal{Q}_α is the set of all probability measures defined on the power set of Ω satisfying $\alpha \mathbb{Q}(\omega) \leq \mathbb{P}(\omega)$ for every $\omega \in \Omega$; see Theorem 4.52 in Föllmer, Schied (2011). Polyhedrality follows from the fact that \mathcal{Q}_α can be expressed as the convex hull of finitely many probability measures. Note that a financial institution would pass this capital adequacy test, roughly speaking, if and only if it is solvent on average over the tail beyond the upper α -quantile level. \square

Another important class of acceptance sets is the one based on Test Scenarios. For instance, the Standard Portfolio ANALysis of Risk (SPAN) methodology, which has been used for decades by the Chicago Mercantile Exchange, is based on such a criterion.

Test Scenarios. The acceptance set based on a nonempty set of *test scenarios* $E \subset \Omega$ is the closed polyhedral cone defined by

$$\mathcal{A}_E := \{X \in \mathcal{X}; X(\omega) \geq 0, \forall \omega \in E\}.$$

In this case, acceptability reduces to requiring solvency in each of the chosen test scenarios. Note that \mathcal{A}_Ω is nothing but the positive cone of \mathcal{X} , i.e. we have $\mathcal{A}_\Omega = \mathcal{X}_+$. \square

Eligible assets

If the capital position of a financial institution is not acceptable, then management will need to undertake remedial actions to reach acceptability. In this paper we assume that the only possible remedial action is to raise new capital and invest it in a portfolio of pre-specified eligible assets. We denote by \mathcal{M} a vector subspace of \mathcal{X} representing the space of *eligible payoffs*, i.e. those random variables that correspond to payoffs of portfolios of eligible assets. We assume that the Law of One Price holds. This implies that every portfolio generating the same payoff has the same initial price so that the map

$$\pi : \mathcal{M} \rightarrow \mathbb{R}$$

assigning to every eligible payoff $Z \in \mathcal{M}$ the initial price $\pi(Z)$ of a portfolio generating it is well defined. Moreover, we assume that the market for eligible assets is frictionless so that π is a linear functional. We do not need to exclude the possibility of arbitrage opportunities but will need to require that \mathcal{M} contains a strictly-positive payoff with strictly-positive price (which can be always normalized to 1), i.e.

(A3) there exists $U \in \mathcal{M}$ such that $U(\omega) > 0$ for all $\omega \in \Omega$ and $\pi(U) = 1$.

For instance, one can assume that U is the payoff of a (default-free) bond or of a stock at time 1.

Finally, we assume that it is not possible to be able to make every position acceptable at zero cost, i.e.

(A4) $\mathcal{A} + \ker(\pi) \neq \mathcal{X}$.

Here, we have denoted by $\ker(\pi)$ the kernel of π , i.e. the subspace of all eligible payoffs having zero price. This requirement is very natural from a regulatory perspective because if it failed to hold, then we would be able to make every unacceptable capital position acceptable by simply adding a zero-cost, i.e. a fully-leveraged, portfolio. In this sense, the above assumption rules out a form of arbitrage that could be called *acceptability arbitrage*.

The optimal payoff map

The *risk measure* associated with the acceptance set \mathcal{A} , the space of eligible payoffs \mathcal{M} , and the pricing functional π is the map $\rho : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ defined by setting

$$\rho(X) := \inf\{\pi(Z); Z \in \mathcal{M}, X + Z \in \mathcal{A}\}.$$

The number $\rho(X)$ represents the “minimum” amount of capital that needs to be raised and invested in some eligible payoff to meet the prescribed acceptability constraint. We write minimum in quotation marks because the infimum in the definition need not be attained. In this sense, $\rho(X)$ can be interpreted as a *capital requirement* or, in the context of internal regulation, as a *margin requirement*. The risk measures introduced by Artzner, Delbaen, Eber, Heath (1999) constitute the prototype of the above capital requirement functionals and correspond to the simplest form of management action, i.e. raising capital and investing it in a single eligible asset. The extension to a multi-asset framework was first studied, to the best of our knowledge, in Föllmer, Schied (2002) and later taken up in Frittelli, Scandolo (2006), Artzner, Delbaen, Koch-Medina (2009) and, more comprehensively, in Farkas, Koch-Medina, Munari (2015).

The *optimal payoff map* associated to the risk measure ρ is the set-valued map $\mathcal{E}_\rho : \mathcal{X} \rightrightarrows \mathcal{M}$ defined by

$$\mathcal{E}_\rho(X) := \{Z \in \mathcal{M}; X + Z \in \mathcal{A}, \rho(X) = \pi(Z)\}.$$

Every eligible payoff in $\mathcal{E}_\rho(X)$ will be called an *optimal payoff* for the position X . Note that, as mentioned above, there may exist capital positions X such that $\mathcal{E}_\rho(X)$ is empty.

The three questions raised in the introduction can now be reformulated in the language of the optimal payoff map.

- **Existence of optimal portfolios.** Do optimal portfolios of eligible assets, i.e. portfolios that allow to pass the regulatory test at the minimum cost, exist at all? This is equivalent to assessing when \mathcal{E}_ρ is not empty valued.
- **Uniqueness of optimal portfolios.** In case optimal portfolios exist, are they unique or does management have several alternatives from which to choose? This is equivalent to assessing when \mathcal{E}_ρ is singleton valued.
- **Stability of optimal portfolios.** In case several optimal portfolios exist, how robust is the choice of a specific portfolio? In other words, if we base our allocation on a slightly misestimated capital position, how confident can we be that this choice will not be “too wrong” for the actual position? This is related to studying the “continuity” of \mathcal{E}_ρ .

As said above, the main objective of this paper is to provide answers to the above questions. The first two questions are clear and are addressed in Section 4. In particular, we provide a variety of necessary and sufficient conditions for both existence and uniqueness. The last question requires specifying what we mean by “continuity” for the set-valued map \mathcal{E}_ρ . This is the content of Section 5.

2.2 Applications beyond capital adequacy

The structure of our capital adequacy problem is intimately linked to the structure of other central problems in finance. We highlight the link to two of these, namely the problem of hedging and providing pricing bounds for payoffs in incomplete markets and the problem of measuring systemic risk.

Pricing and hedging with basis risk

We interpret the elements of \mathcal{X} as payoffs of financial contracts and we take \mathcal{M} to be the subspace of all marketed payoffs, i.e. those payoffs that can be perfectly replicated by a suitable portfolio of traded assets. We assume the market is incomplete so that \mathcal{M} is a strict subspace of \mathcal{X} .

Assume an agent has “sold” a financial contract with payoff $X \in \mathcal{X}$, which he or she has to honour at time 1. A marketed payoff $Z \in \mathcal{M}$ is said to be a *hedge* for X if $Z \geq X$, i.e. if buying Z allows the agent to meet all obligations related to X in all states of the economy. Note that this can be expressed as

$$Z - X \in \mathcal{X}_+.$$

More generally, one may consider hedging with basis risk, i.e. hedging with a certain tolerance for error. Here, instead of taking \mathcal{X}_+ we take a general acceptance set \mathcal{A} which reflects our error tolerance. In this context Z is an \mathcal{A} -*hedge* for X if the hedging error $Z - X$ belongs to \mathcal{A} , i.e.

$$Z - X \in \mathcal{A}.$$

We single out two specific types of acceptability criteria for the hedging error.

Hedging based on VaR. Assume that we assign to each state of the economy a probability of occurrence so that a probability measure \mathbb{P} is defined on the power set of Ω . The acceptance set based on VaR at the level $\alpha \in (0, 1)$

$$\mathcal{A}_{\text{VaR}}(\alpha) = \{X \in \mathcal{X}; \mathbb{P}(X < 0) \leq \alpha\}$$

corresponds to allowing for a negative hedging error in at most $100\alpha\%$ of the cases. \square

Hedging based on Shortfall Risk. Let \mathbb{P} be as above. Let $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. For a point $\alpha \in \mathbb{R}$ in the interior of the range of ℓ define the closed and convex acceptance set

$$\mathcal{A}_\ell(\alpha) := \{X \in \mathcal{X}; \mathbb{E}_\mathbb{P}[\ell(\max\{-X, 0\})] \leq \alpha\}.$$

The *loss function* ℓ represents the way an agent assigns a degree of “disutility” to negative hedging errors according to their size. Using this acceptance set corresponds to allowing a hedging error with a maximal expected disutility of α . \square

The minimum \mathcal{A} -hedging cost for a payoff $X \in \mathcal{X}$ is then given by

$$\rho(-X) = \inf\{\pi(Z); Z \in \mathcal{M}, Z - X \in \mathcal{A}\}.$$

Hence, the minimum \mathcal{A} -hedging cost is obtained by applying the risk measure ρ associated with the acceptance set \mathcal{A} , the space of eligible payoffs \mathcal{M} , and the pricing functional π to the payoff $-X$. In a pricing framework, the above quantity can be interpreted as a *price from a seller's perspective*. Similarly, the quantity

$$-\rho(X) = \sup\{\pi(Z); Z \in \mathcal{M}, X - Z \in \mathcal{A}\}$$

can be interpreted as a *price from a buyer's perspective*. The above amounts provide an upper, respectively a lower, bound for a transaction to take place between two agents that have the same tolerance for risk captured by \mathcal{A} . If we allow for no hedging error, i.e. if $\mathcal{A} = \mathcal{X}_+$, then we obtain the standard *superhedging* and *subhedging prices*. If \mathcal{A} is larger than \mathcal{X}_+ , thus allowing for hedging with basis risk, we are in the framework of *good deal prices*.

The set of optimal \mathcal{A} -hedging payoffs for a payoff $X \in \mathcal{X}$ corresponds to

$$\mathcal{E}_\rho(-X) = \{Z \in \mathcal{M}; Z - X \in \mathcal{A}, \pi(Z) = \rho(-X)\}.$$

In a pricing framework this is the set of \mathcal{A} -hedging payoffs whose price equals the associated good deal price from a seller's perspective. The corresponding set of optimal payoffs from a buyer's perspective is given by

$$-\mathcal{E}_\rho(X) = \{Z \in \mathcal{M}; X - Z \in \mathcal{A}, \pi(Z) = -\rho(X)\}.$$

For a review of the theory of *good deal pricing* or *pricing under acceptable risk* we refer to the original contributions by Cochrane, Saa-Requejo (2000), by Carr, Geman, Madan (2001), and by Jaschke, Küchler (2001). This pricing approach was recently revived in the framework of *conic finance* starting with Madan, Cherny (2010). Indifference pricing with acceptable risk is discussed in Arai, Fukasawa (2014).

Capital allocation and systemic risk

Let $d \in \mathbb{N}$ and consider d financial institutions with respective capital positions $X_1, \dots, X_d \in \mathcal{X}$. The random vector

$$(X_1, \dots, X_d) \in \mathcal{X}^d$$

represents the *system* of the d entities. In this context, an acceptance set $\mathcal{C} \subset \mathcal{X}^d$ aims to capture the adequacy of the system as a whole. If \mathcal{C} is of the form

$$\mathcal{C} = \mathcal{A}_1 \times \dots \times \mathcal{A}_d$$

where each component is an acceptance set in \mathcal{X} , then this amounts to declaring the system adequately capitalized if every entity is adequately capitalized on a stand-alone basis according to the corresponding univariate acceptance sets. In this case, the interdependencies between entities would not be taken into consideration when assessing the adequacy of the system. More relevant, then, are acceptance sets where the interdependencies do matter. This is the case if there exists an *aggregation function* $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\mathcal{C} = \{(X_1, \dots, X_d) \in \mathcal{X}^d ; \Lambda(X_1, \dots, X_d) \in \mathcal{A}\}$$

where \mathcal{A} is a given acceptance set in \mathcal{X} . For every random vector $(X_1, \dots, X_d) \in \mathcal{X}^d$ the quantity

$$\rho(X_1, \dots, X_d) = \inf \left\{ \sum_{i=1}^d \pi(Z_i) ; (Z_1, \dots, Z_d) \in \mathcal{M}^d, (X_1 + Z_1, \dots, X_d + Z_d) \in \mathcal{C} \right\}$$

can be interpreted as the “minimum” amount of capital that has to be raised for the entire system and allocated, in the form of optimal portfolios of eligible assets, across the various entities of the system to ensure acceptability. Note that, in principle, one could allow for different spaces of eligible assets and, hence, different pricing functionals across the different components.

The set of vectors of eligible payoffs that ensure the acceptability of the system $(X_1, \dots, X_d) \in \mathcal{X}^d$ at the lowest cost is then given by

$$\mathcal{E}_\rho(X_1, \dots, X_d) = \left\{ (Z_1, \dots, Z_d) \in \mathcal{M}^d ; (X_1 + Z_1, \dots, X_d + Z_d) \in \mathcal{C}, \rho(X_1, \dots, X_d) = \sum_{i=1}^d \pi(Z_i) \right\}.$$

Also here it is critical to study the existence and multiplicity of optimal vectors of eligible payoffs and the continuity properties of the resulting set-valued map. Even though our results are formulated in the context of the space \mathcal{X} , i.e. the space of univariate random variables over the finite state space Ω , they can be easily extended to any finite-dimensional topological vector space equipped with a partial order such as the space \mathcal{X}^d . Alternatively, it is not difficult to convert the above multivariate setting into our scalar setting by using the fact that any random vector $(X_1, \dots, X_d) \in \mathcal{X}^d$ can be identified with a random variable

$$X : \Omega \times \{1, \dots, d\} \rightarrow \mathbb{R}$$

defined by setting $X(\omega, i) = X_i(\omega)$ for every $\omega \in \Omega$ and $i \in \{1, \dots, d\}$.

The above *capital allocation functional* is a general example of the systemic risk measures defined in Biagini, Fouque, Frittelli, Meyer-Brandis (2016). The special case corresponding to acceptability based on multivariate shortfall risk is studied in Armenti, Crépey, Drapeau, Papapantoleon (2016), where the problem of existence and uniqueness is explicitly posed and tackled. The corresponding set-valued maps are closely related to the so-called efficient cash-invariant allocation rules, which have been recently introduced in the framework of systemic risk measures in Feinstein, Rudloff, Weber (2016).

3 Basic properties of the optimal payoff map

In this section we collect a variety of basic properties of the optimal payoff map that will be freely used in the remainder of the paper.

We start by listing some important properties of the risk measure ρ . We refer to Farkas, Koch-Medina, Munari (2015) for a proof.

Proposition 3.1. *The risk measure ρ is finitely valued, (Lipschitz) continuous and for every $X, Y \in \mathcal{X}$ satisfies:*

- (i) $X \geq Y$ implies $\rho(X) \leq \rho(Y)$.
- (ii) $\rho(X + Z) = \rho(X) - \pi(Z)$ for every $Z \in \mathcal{M}$.
- (iii) $\rho(X) = \inf\{m \in \mathbb{R}; X + mU \in \mathcal{A} + \ker(\pi)\}$.

Remark 3.2. It follows from point (iii) above that assumption (A4) is necessary to ensure that ρ is finitely valued. Under this assumption, (A3) implies that ρ is finitely valued and continuous by Proposition 1 in Farkas, Koch-Medina, Munari (2015). \square

We start by establishing a useful reformulation of the set of optimal payoffs. To this effect, we will use the next lemma, which is a direct consequence of Proposition 3.1.

Lemma 3.3. *The zero level sets of ρ can be obtained as follows:*

- (i) $\{X \in \mathcal{X}; \rho(X) < 0\} = \text{int}(\mathcal{A} + \ker(\pi))$.
- (ii) $\{X \in \mathcal{X}; \rho(X) \leq 0\} = \text{cl}(\mathcal{A} + \ker(\pi))$.
- (iii) $\{X \in \mathcal{X}; \rho(X) = 0\} = \text{bd}(\mathcal{A} + \ker(\pi))$.

Proposition 3.4. *For every $X \in \mathcal{X}$ the set $\mathcal{E}_\rho(X)$ admits the representation*

$$\mathcal{E}_\rho(X) = \{Z \in \mathcal{M}; X + Z \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))\}.$$

In particular, $\mathcal{E}_\rho(X)$ is closed for every $X \in \mathcal{X}$.

Proof. Take any $Z \in \mathcal{E}_\rho(X)$ and note that, by definition, we have $X + Z \in \mathcal{A}$ and $\rho(X) = \pi(Z)$. Should $X + Z \in \text{int}(\mathcal{A})$ hold, we would find a suitable $\varepsilon > 0$ such that $X + Z - \varepsilon U \in \mathcal{A}$. However, this would imply that

$$\rho(X) \leq \pi(Z) - \varepsilon \pi(U) < \rho(X).$$

Hence, we must have $X + Z \in \text{bd } \mathcal{A}$. Moreover, $\rho(X) = \pi(Z)$ implies that $\rho(X + Z) = 0$ and therefore $X + Z \in \text{bd}(\mathcal{A} + \ker(\pi))$ by Lemma 3.3. This establishes the inclusion “ \subset ”. Conversely, take $Z \in \mathcal{M}$ satisfying $X + Z \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$. Since \mathcal{A} is closed, we easily see that $X + Z \in \mathcal{A}$. Moreover, $X + Z \in \text{bd}(\mathcal{A} + \ker(\pi))$ implies that $\rho(X + Z) = 0$ by Lemma 3.3, which in turn entails $\rho(X) = \pi(Z)$. Hence, it follows that $Z \in \mathcal{E}_\rho(X)$ and the inclusion “ \supset ” is also established. The closedness of $\mathcal{E}_\rho(X)$ is a direct consequence of the above representation. \square

The next proposition lists some useful properties of the optimal payoff map, which will be freely used without reference in the sequel. For the sake of completeness, we provide explicit proofs of all of them. Here, we denote by \mathcal{C}^∞ the asymptotic cone of a set $\mathcal{C} \subset \mathcal{X}$; see the appendix for more details.

Proposition 3.5. *The following statements hold for every $X \in \mathcal{X}$:*

- (i) $\mathcal{E}_\rho(X + Z) = \mathcal{E}_\rho(X) - Z$ for every $Z \in \mathcal{M}$.
- (ii) $\mathcal{E}_\rho(\mathcal{K})$ is closed for every compact set $\mathcal{K} \subset \mathcal{X}$.
- (iii) $\mathcal{E}_\rho(X)$ is convex if \mathcal{A} is convex.
- (iv) $\mathcal{E}_\rho(X)$ is polyhedral if \mathcal{A} is polyhedral.
- (v) $\mathcal{E}_\rho(X)^\infty \subset \mathcal{A}^\infty \cap \ker(\pi)$.
- (vi) $\mathcal{E}_\rho(X)^\infty = \mathcal{A}^\infty \cap \ker(\pi)$ if \mathcal{A} is star shaped and $\mathcal{E}_\rho(X) \neq \emptyset$.

Proof. (i) Fix $Z \in \mathcal{M}$ and take any $W \in \mathcal{E}_\rho(X + Z)$. Then, we have $X + W + Z \in \mathcal{A}$ as well as

$$\pi(W) = \rho(X + Z) = \rho(X) - \pi(Z),$$

so that $\pi(W + Z) = \rho(X)$. This shows that $W + Z \in \mathcal{E}_\rho(X)$ and, hence, $\mathcal{E}_\rho(X + Z) \subset \mathcal{E}_\rho(X) - Z$ holds. To prove the converse inclusion, take any $W \in \mathcal{E}_\rho(X)$ and note that

$$X + Z + W - Z = X + W \in \mathcal{A}.$$

Moreover, we also easily see that

$$\pi(W - Z) = \rho(X) - \pi(Z) = \rho(X + Z).$$

As a result, it follows that $W - Z \in \mathcal{E}_\rho(X + Z)$ so that $\mathcal{E}_\rho(X) - Z \subset \mathcal{E}_\rho(X + Z)$ holds. This concludes the proof of (i).

(ii) Assume that $\mathcal{K} \subset \mathcal{X}$ is compact and consider a sequence $(Z_n) \subset \mathcal{E}_\rho(\mathcal{K})$ converging to some $Z \in \mathcal{M}$. Without loss of generality we can find a sequence $(X_n) \subset \mathcal{K}$ converging to a suitable $X \in \mathcal{K}$ and such that $Z_n \in \mathcal{E}_\rho(X_n)$ for every $n \in \mathbb{N}$. We claim that $Z \in \mathcal{E}_\rho(X)$. To see this, note first that $X_n + Z_n \in \mathcal{A}$ for every $n \in \mathbb{N}$ and $X_n + Z_n \rightarrow X + Z$, so that $X + Z \in \mathcal{A}$. In addition, we have

$$\pi(Z) = \lim_{n \rightarrow \infty} \pi(Z_n) = \lim_{n \rightarrow \infty} \rho(X_n) = \rho(X)$$

by the continuity of ρ . This establishes that $Z \in \mathcal{E}_\rho(X)$ and concludes the proof of (ii).

(iii) Assume first that \mathcal{A} is convex and take any $X \in \mathcal{X}$. Then, for every $Z, W \in \mathcal{E}_\rho(X)$ and for every $\lambda \in [0, 1]$ we clearly have

$$X + \lambda Z + (1 - \lambda)W = \lambda(X + Z) + (1 - \lambda)(X + W) \in \mathcal{A}$$

as well as

$$\pi(\lambda Z + (1 - \lambda)W) = \lambda\rho(X) + (1 - \lambda)\rho(X) = \rho(X).$$

This shows that $\mathcal{E}_\rho(X)$ is convex and establishes (iii).

(iv) Assume that \mathcal{A} is polyhedral so that we find suitable linear functionals $\psi_1, \dots, \psi_m : \mathcal{X} \rightarrow \mathbb{R}$ and scalars $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ satisfying

$$\mathcal{A} = \bigcap_{i=1}^m \{X \in \mathcal{X}; \psi_i(X) \geq \alpha_i\}.$$

Then, it is easy to see that

$$\mathcal{E}_\rho(X) = \bigcap_{i=1}^m \{Z \in \mathcal{M}; \psi_i(Z) \geq \alpha_i - \psi(X)\} \cap \{Z \in \mathcal{M}; \pi(Z) \geq \rho(X), -\pi(Z) \geq -\rho(X)\} \quad (1)$$

for every $X \in \mathcal{X}$, showing that $\mathcal{E}_\rho(X)$ is a polyhedral set in \mathcal{M} . This establishes (iv).

(v) Take any $Z \in \mathcal{E}_\rho(X)^\infty$ so that $\lambda_n Z_n \rightarrow Z$ for a suitable sequence $(\lambda_n) \subset \mathbb{R}_+$ converging to zero and $(Z_n) \subset \mathcal{E}_\rho(X)$. Since we have $\lambda_n(X + Z_n) \rightarrow Z$ and $X + Z_n \in \mathcal{A}$ for every $n \in \mathbb{N}$, we see that $Z \in \mathcal{A}^\infty$. Moreover, note that Z belongs to \mathcal{M} and satisfies

$$\pi(Z) = \lim_{n \rightarrow \infty} \lambda_n \pi(Z_n) = \lim_{n \rightarrow \infty} \lambda_n \rho(X) = 0,$$

so that $Z \in \ker(\pi)$. This proves (v).

(vi) Recall that, if \mathcal{A} is star shaped, we have $\mathcal{A}^\infty = \text{rec } \mathcal{A}$. Moreover, recall that asymptotic cones always contain the corresponding recession cones. Hence, in light of point (v), the claim will be established once we prove that

$$\text{rec } \mathcal{A} \cap \ker(\pi) \subset \text{rec } \mathcal{E}_\rho(X). \quad (2)$$

To this effect, take any $Z \in \text{rec } \mathcal{A} \cap \ker(\pi)$ and $W \in \mathcal{E}_\rho(X)$, which exists since $\mathcal{E}_\rho(X)$ is assumed to be nonempty. We claim that, for every $\lambda \in (0, \infty)$, we have $W + \lambda Z \in \mathcal{E}_\rho(X)$. To show this, note first that $X + W + \lambda Z \in \mathcal{A}$. This follows from the fact that $Z \in \text{rec } \mathcal{A}$ and $X + W \in \mathcal{A}$. Moreover, it is clear that

$$\pi(W + \lambda Z) = \pi(W) = \rho(X).$$

This shows that $X + W + \lambda Z \in \mathcal{A}$ for every $\lambda \in (0, \infty)$ and establishes (2). □

Remark 3.6. We show that the assumptions in point (vi) above are all necessary to ensure that $\mathcal{E}_\rho(X)^\infty = \mathcal{A}^\infty \cap \ker(\pi)$ holds. Let $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{M} = \mathcal{X}$ and define the pricing functional π by setting $\pi(X) = \frac{1}{2}(X(\omega_1) + X(\omega_2))$ for all $X \in \mathcal{X}$.

(i) \mathcal{A} is star shaped but $\mathcal{E}_\rho(X)$ is empty. Consider the acceptance set $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where

$$\mathcal{A}_1 = \{X \in \mathcal{X}; X(\omega_1) \in [0, \infty), X(\omega_2) \in [-1, \infty)\}$$

and

$$\mathcal{A}_2 = \{X \in \mathcal{X}; X(\omega_1) \in (-\infty, 0), X(\omega_2) \geq e^{X(\omega_1)} - X(\omega_1) - 2\}.$$

Note that \mathcal{A} is star shaped. It is easy to verify that our assumptions (A1) to (A4) are all satisfied in this setting. Moreover, we have

$$\mathcal{A} + \ker(\pi) = \{X \in \mathcal{X}; X(\omega_2) > -X(\omega_1) - 2\}.$$

Since $\text{bd } \mathcal{A}$ and $\text{bd}(\mathcal{A} + \ker(\pi))$ have empty intersection, it follows from Proposition 3.4 that $\mathcal{E}_\rho(X)$ is empty for every $X \in \mathcal{X}$. However, $\mathcal{A}^\infty \cap \ker(\pi)$ contains infinitely many elements.

(ii) $\mathcal{E}_\rho(X)$ is nonempty but \mathcal{A} is not star shaped. Set $\alpha_n = -n + \frac{1}{n}$ for every $n \in \mathbb{N}$ and consider the acceptance set

$$\mathcal{A} = \mathcal{X}_+ \cup \bigcup_{n \in \mathbb{N}} \{X \in \mathcal{X}; X(\omega_1) \in [\alpha_{n+1}, \alpha_n), X(\omega_2) \in [n, \infty)\}.$$

Note that \mathcal{A} is not star shaped and that our assumptions (A1) to (A4) are all satisfied in this setting. Moreover, it is easy to verify that

$$\mathcal{A} + \ker(\pi) = \{X \in \mathcal{X}; X(\omega_2) \geq -X(\omega_1)\}.$$

Since $\mathcal{E}_\rho(0) = \{0\}$, we also have $\mathcal{E}_\rho(0)^\infty = \{0\}$. However, $\mathcal{A}^\infty \cap \ker(\pi)$ is easily seen to contain infinitely many elements. \square

4 Existence and uniqueness of optimal payoffs

This section is devoted to investigating whether and under which conditions we can ensure the existence and the uniqueness of optimal eligible payoffs, i.e. eligible payoffs that allow to reach acceptability at the minimal cost.

Existence of optimal payoffs

We start by providing a general characterization of the existence of optimal payoffs for every capital position.

Proposition 4.1. *The following statements are equivalent:*

- (a) $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$.
- (b) $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \text{bd}(\mathcal{A} + \ker(\pi))$.
- (c) $\mathcal{A} + \ker(\pi)$ is closed.

Proof. It is clear that (a) implies (b). Assume now that (b) holds but $\mathcal{A} + \ker(\pi)$ is not closed, so that we find $X \in \text{bd}(\mathcal{A} + \ker(\pi)) \setminus (\mathcal{A} + \ker(\pi))$. This implies that $\mathcal{E}_\rho(X) \neq \emptyset$ and $\rho(X) = 0$ by Lemma 3.3 but, at the same time, that there cannot exist $Z \in \ker(\pi)$ with $X + Z \in \mathcal{A}$. Since this is not possible, we conclude that (c) must hold.

Finally, assume that (c) holds and take an arbitrary $X \in \mathcal{X}$. Since we clearly have $\rho(X + \rho(X)U) = 0$, it follows from Lemma 3.3 that $X + \rho(X)U \in \mathcal{A} + \ker(\pi)$ and this yields $\mathcal{E}_\rho(X) \neq \emptyset$. This shows that (a) holds and concludes the proof. \square

The preceding result shows that we cannot ensure the existence of optimal payoffs for every position unless the set $\mathcal{A} + \ker(\pi)$ is closed. Establishing the closedness of the sum of two closed sets has been a classical problem in the theory of topological vector spaces since the early publication of Dieudonné (1966), where the notion of asymptotic cone was first used to tackle the closedness problem. The following sufficient condition for the existence of optimal payoffs is a direct consequence of a suitable generalization of Dieudonné's criterion recorded in Theorem 2.3.4 in Auslender, Teboulle (2003).

Proposition 4.2. *Assume that $\mathcal{A}^\infty \cap \ker(\pi) = \{0\}$. Then, $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$.*

Proof. Note that the condition $\mathcal{A}^\infty \cap \ker(\pi) = \{0\}$ implies that we must have $X = Y = 0$ whenever $X \in \mathcal{A}^\infty$ and $Y \in \ker(\pi)$. Hence, in the language of Definition 2.3.3 in Auslender, Teboulle (2003), the sets \mathcal{A} and $\ker(\pi)$ are in general position. As a result, it follows from Theorem 2.3.4 in the aforementioned book that $\mathcal{A} + \ker(\pi)$ is closed and, thus, $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$ by virtue of the above proposition. \square

Remark 4.3. In the setting of multivariate shortfall risk measures studied by Armenti, Crépey, Drapeau, Papapantoleon (2016) the existence of optimal payoffs, which are called *risk allocations* in their paper, is a crucial question to define a meaningful allocation of systemic risk across the entities of the financial system under inspection. The main result in this respect is their Theorem 3.6, which provides a sufficient condition for the existence of optimal payoffs under a suitable assumption on the underlying multivariate loss function. This assumption is easily seen to be equivalent to the sufficient condition in the above proposition. The elements of $\ker(\pi)$ are called *zero-sum allocations* there. \square

Since every polyhedral set is closed and the sum of two polyhedral sets is still polyhedral, it follows that every position admits an optimal payoff whenever the chosen acceptance set is polyhedral.

Corollary 4.4. *Assume that \mathcal{A} is polyhedral. Then, $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$.*

We conclude by highlighting that, since any star shaped set that is closed contains its asymptotic cone, we always find optimal payoffs for any position provided that \mathcal{A} is star shaped and the only eligible payoff with zero cost that is acceptable is the null payoff.

Corollary 4.5. *Assume that \mathcal{A} is star shaped and $\mathcal{A} \cap \ker(\pi) = \{0\}$. Then, we have $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$.*

Remark 4.6. In the case that $\mathcal{A} = \mathcal{X}_+$, the condition $\mathcal{A} \cap \ker(\pi) = \{0\}$ stipulates the absence of arbitrage opportunities. Hence, requiring $\mathcal{A} \cap \ker(\pi) = \{0\}$ can be interpreted as ruling out a generalized form of arbitrage opportunities, which are often referred to as *good deals* in the literature. We refer to Section 2.2 for a variety of references about good deal pricing. \square

Uniqueness of optimal payoffs

After characterizing the existence of optimal payoffs, the next natural question is under which conditions we can ensure their uniqueness. We prove two characterizations of uniqueness. The first result is a simple consequence of the definition of our optimal payoff map. Here, we denote by $|\mathcal{C}|$ the cardinality of any set $\mathcal{C} \subset \mathcal{X}$.

Proposition 4.7. *Assume that $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$. Then, the following statements are equivalent:*

- (a) $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \mathcal{X}$.
- (b) $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \text{bd } \mathcal{A} \cap \text{bd } (\mathcal{A} + \ker(\pi))$.
- (c) $\text{bd } \mathcal{A} \cap \text{bd } (\mathcal{A} + \ker(\pi)) \cap (\text{bd } \mathcal{A} + (\ker(\pi) \setminus \{0\})) = \emptyset$.
- (d) $\text{bd } \mathcal{A} \cap (\text{bd } \mathcal{A} + (\ker(\pi) \setminus \{0\})) \subset \text{int}(\mathcal{A} + \ker(\pi))$.

Proof. It is clear that (a) implies (b). Now, assume that (b) holds but we find $X \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$ and $Z \in \ker(\pi) \setminus \{0\}$ such that $X + Z \in \text{bd } \mathcal{A}$. Since $\rho(X) = 0$ by Lemma 3.3, we see that $\mathcal{E}_\rho(X)$ contains both the null payoff $0 \in \mathcal{M}$ and the nonzero payoff $Z \in \mathcal{M}$ so that $|\mathcal{E}_\rho(X)| \geq 2$. Since this contradicts (b), we conclude that (b) must imply (c).

It is immediate to verify that (c) implies (d). Finally, assume that condition (d) is satisfied but there exist $Z_1, Z_2 \in \mathcal{E}_\rho(X)$ with $Z_1 \neq Z_2$ for some $X \in \mathcal{X}$. In particular, note that $Z_2 - Z_1 \in \ker(\pi) \setminus \{0\}$. Since Proposition 3.4 implies that $X + Z_1 \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$, it follows that

$$X + Z_2 = X + Z_1 + Z_2 - Z_1 \in ((\text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))) + \ker(\pi) \setminus \{0\}) \cap \text{bd } \mathcal{A}.$$

However, this is incompatible with condition (d), showing that (d) implies (a) and concluding the proof of the proposition. \square

We provide sufficient conditions for uniqueness in the case of a polyhedral acceptance set. We refer to Proposition A.2 for a variety of properties of polyhedral sets and their canonical representation. For a polyhedral set $\mathcal{C} \subset \mathcal{X}$ represented by the functionals $\psi_1, \dots, \psi_m \in \mathcal{X}'$ we define for any $X \in \mathcal{X}$ the set of *active constraints* by setting

$$I_a(X) := \{i \in \{1, \dots, m\}; \psi_i(X) = \sigma_{\mathcal{C}}(\psi_i)\}.$$

The (lower) support function $\sigma_{\mathcal{C}}$ is defined in the appendix. Recall that $I_a(X)$ is nonempty if, and only if, X is a boundary point of \mathcal{C} .

Proposition 4.8. *Assume \mathcal{A} is polyhedral and is represented by $\psi_1, \dots, \psi_m \in \mathcal{X}'_+$. Then, the following statements are equivalent:*

- (a) $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \mathcal{X}$.
- (b) $\ker(\pi) \cap \bigcap_{i \in I_a(X)} \ker(\psi_i) = \{0\}$ for every $X \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$.

Proof. First of all, recall from Corollary 4.4 that every position admits an optimal payoff so that $\mathcal{E}_\rho(X)$ is nonempty for all $X \in \mathcal{X}$. To prove that (a) implies (b), assume that condition (b) fails for $X \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$ so that we find a nonzero $Z \in \ker(\pi)$ that belongs to $\ker(\psi_i)$ for all $i \in I_a(X)$. In particular, note that

$$\psi_i(X + \lambda Z) = \psi_i(X) + \lambda \psi_i(Z) = \sigma_{\mathcal{A}}(\psi_i) \quad \text{for } i \in I_a(X)$$

for every $\lambda \in (0, \infty)$. Since $\psi_i(X) > \sigma_{\mathcal{A}}(\psi_i)$ for $i \notin I_a(X)$, it is also clear that

$$\psi_i(X + \lambda Z) = \psi_i(X) + \lambda \psi_i(Z) \geq \sigma_{\mathcal{A}}(\psi_i) \quad \text{for } i \notin I_a(X)$$

for $\lambda \in (0, \infty)$ small enough. This implies that $X + \lambda Z \in \mathcal{A}$ for $\lambda \in (0, \infty)$ small enough. Since $\rho(X) = 0$, we conclude that $|\mathcal{E}_\rho(X)| > 1$. This establishes that (a) implies (b).

Conversely, assume that (a) does not hold so that $|\mathcal{E}_\rho(X)| > 1$ for some $X \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$ by Proposition 4.7. Then, we find distinct $Z_1, Z_2 \in \mathcal{M}$ such that $X + Z_1$ and $X + Z_2$ both belong to \mathcal{A} and $\pi(Z_1) = \rho(X) = \pi(Z_2)$. In particular, note that $Z_1 - Z_2 \in \ker(\pi) \setminus \{0\}$. Now, set $Y = X + \frac{1}{2}(Z_1 + Z_2) \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$. Then, for every $i \in I_a(Y)$ we have

$$\sigma_{\mathcal{A}}(\psi_i) + \frac{1}{2}\psi_i(Z_1 - Z_2) = \psi_i(Y) + \frac{1}{2}\psi_i(Z_1 - Z_2) = \psi_i(X + Z_1) \geq \sigma_{\mathcal{A}}(\psi_i)$$

as well as

$$\sigma_{\mathcal{A}}(\psi_i) + \frac{1}{2}\psi_i(Z_2 - Z_1) = \psi_i(Y) + \frac{1}{2}\psi_i(Z_2 - Z_1) = \psi_i(X + Z_2) \geq \sigma_{\mathcal{A}}(\psi_i).$$

This implies that $\psi_i(Z_1 - Z_2) = 0$ for all $i \in I_a(Y)$, which shows that (b) is violated. It follows that (b) implies (a). \square

As a corollary, we immediately obtain the following simple sufficient condition for uniqueness for polyhedral acceptance sets that are also cones, i.e. for coherent polyhedral acceptance sets.

Corollary 4.9. Assume \mathcal{A} is a polyhedral cone represented by $\psi_1, \dots, \psi_m \in \mathcal{X}'_+$ and assume that

$$\ker(\pi) \cap \ker(\psi_i) = \{0\} \quad \text{for all } i \in \{1, \dots, m\}.$$

Then, we have $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \mathcal{X}$.

We conclude this section by establishing a useful sufficient condition for uniqueness, which will be applied to strictly-convex acceptance sets below.

Proposition 4.10. Assume that $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$ and that for every distinct $X, Y \in \text{bd } \mathcal{A}$ such that $X - Y \in \ker(\pi)$ there exists $\lambda \in (0, 1)$ satisfying

$$\lambda X + (1 - \lambda)Y \in \text{int } \mathcal{A}.$$

Then, $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \mathcal{X}$.

Proof. Take any position $X \in \text{bd } \mathcal{A} \cap (\text{bd } \mathcal{A} + \ker(\pi) \setminus \{0\})$. The claim will follow directly from Proposition 4.7 once we show that X belongs to $\text{int}(\mathcal{A} + \ker(\pi))$. To this effect, note that $X = Y + Z$ for a suitable position $Y \in \text{bd } \mathcal{A}$ and a nonzero payoff $Z \in \ker(\pi)$. Hence, by assumption, we find a scalar $\lambda \in (0, 1)$ satisfying $\lambda X + (1 - \lambda)Y \in \text{int } \mathcal{A}$. Now, set for convenience

$$M = X - (1 - \lambda)Z = \lambda X + (1 - \lambda)Y \in \text{int } \mathcal{A},$$

so that $\mathcal{B}_r(M) \subset \mathcal{A}$ for some $r > 0$. Here, $\mathcal{B}_r(M)$ stands for the closed ball of radius r centered at M . Since every $W \in \mathcal{B}_r(X)$ is easily seen to satisfy

$$W - (1 - \lambda)Z - M = W - X,$$

it follows that

$$\mathcal{B}_r(X) \subset \mathcal{B}_r(M) + (1 - \lambda)Z \subset \mathcal{A} + \ker(\pi)$$

and, thus, X is an interior point of $\mathcal{A} + \ker(\pi)$. □

Remark 4.11. The above sufficient condition for uniqueness is not necessary in general. To see this, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the polyhedral acceptance set given by

$$\mathcal{A} = \text{co}(\{X_1, X_2, X_3\}) + \mathcal{X}_+$$

where $X_1 = -1_{\omega_2} + 1_{\omega_3}$, $X_2 = -1_{\omega_1} + 1_{\omega_3}$, and $X_3 = -\frac{1}{2}1_{\omega_1}$. Assume that $\mathcal{M} = \mathcal{X}$ and define π by setting $\pi(X) = \frac{1}{3}(X(\omega_1) + X(\omega_2) + X(\omega_3))$ for all $X \in \mathcal{X}$. It is immediate to verify that our assumptions (A1) to (A4) are all satisfied in this setting. Moreover, since

$$\mathcal{A} + \ker(\pi) = \{X \in \mathcal{X}; X(\omega_1) + X(\omega_2) + X(\omega_3) \geq -\frac{1}{2}\},$$

it follows that, for every $X \in \mathcal{X}$, the set $\mathcal{E}_\rho(X)$ is nonempty by Proposition 4.1 and is easily seen to consist of a single payoff due to Proposition 4.7. However, the positions X_1 and X_2 both belong to $\text{bd } \mathcal{A}$ and satisfy $X_1 - X_2 \in \ker(\pi) \setminus \{0\}$ and the entire segment connecting X_1 and X_2 lie in $\text{bd } \mathcal{A}$. □

The following uniqueness result in the case of a strictly-convex acceptance set follows immediately from the above result. Here, we say that $\mathcal{C} \subset \mathcal{X}$ is *strictly convex* whenever \mathcal{C} is convex and $\lambda X + (1 - \lambda)Y \in \text{int } \mathcal{C}$ for every $\lambda \in (0, 1)$ and every $X, Y \in \text{bd } \mathcal{C}$.

Corollary 4.12. Assume that $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$ and \mathcal{A} is strictly convex. Then, we have $|\mathcal{E}_\rho(X)| = 1$ for every $X \in \mathcal{X}$.

Remark 4.13. In the setting of multivariate shortfall risk measures studied by Armenti, Crépey, Drapeau, Papapantoleon (2016) the uniqueness of optimal payoffs, which are called *risk allocations* there, is a crucial property to define a meaningful allocation of systemic risk across the entities of the financial system under investigation. The key uniqueness result is recorded in their Theorem 3.6 and boils down to requiring the strict convexity of their underlying acceptance set. □

5 Stability of optimal payoffs

The remaining part of the paper addresses our third motivating question, which is concerned with the stability or robustness of the optimal portfolio selection. From an operational perspective it is of cardinal importance to ensure that a slight *perturbation* of X , which could be induced by estimation or specification errors, does not alter the set of optimal payoffs in a significant way. In other words, for every positions $X, Y \in \mathcal{X}$ we would like to ensure that

$$Y \text{ is close to } X \implies \mathcal{E}_\rho(Y) \text{ is "close" to } \mathcal{E}_\rho(X).$$

In order to specify a notion of proximity between the sets of optimal payoffs we are naturally led to investigate the (semi)continuity properties of the optimal payoff map.

A variety of (semi)continuity notions for set-valued maps has been investigated in the literature and the same terminology is often used to capture different forms of continuity. Here, we adopt the language of Rockafellar, Wets (2009) and focus on outer semicontinuity, upper semicontinuity, and inner or lower semicontinuity. We refer to the commentary at the end of Chapter 5 in the above-cited book for a discussion about these continuity notions.

Outer semicontinuity. For any given position $X \in \mathcal{X}$ we say that \mathcal{E}_ρ is *outer semicontinuous at X* if for every $Z \notin \mathcal{E}_\rho(X)$ we find open neighborhoods \mathcal{U}_X of X and \mathcal{U}_Z of Z such that

$$Y \in \mathcal{U}_X \implies \mathcal{E}_\rho(Y) \cap \mathcal{U}_Z = \emptyset.$$

This stability property essentially amounts to requiring that a non-optimal marketed payoff cannot become optimal by means of a slight perturbation.

Upper semicontinuity. For any given position $X \in \mathcal{X}$ we say that \mathcal{E}_ρ is *upper semicontinuous at X* if for every open set $\mathcal{U} \subset \mathcal{X}$ with $\mathcal{E}_\rho(X) \subset \mathcal{U}$ we find an open neighborhood \mathcal{U}_X of X such that

$$Y \in \mathcal{U}_X \implies \mathcal{E}_\rho(Y) \subset \mathcal{U}.$$

Since \mathcal{E}_ρ is closed valued, it is immediate to see that upper semicontinuity is a stronger property than outer semicontinuity. Intuitively speaking, upper semicontinuity ensures that the set of optimal payoffs does not suddenly “explode” as a result of a slight perturbation of the underlying position.

Inner semicontinuity. For a position $X \in \mathcal{X}$ we say that \mathcal{E}_ρ is *inner semicontinuous at X* if for every open set $\mathcal{U} \subset \mathcal{X}$ with $\mathcal{E}_\rho(X) \cap \mathcal{U} \neq \emptyset$ we find an open neighborhood \mathcal{U}_X of X such that

$$Y \in \mathcal{U}_X \implies \mathcal{E}_\rho(Y) \cap \mathcal{U} \neq \emptyset.$$

In this case, one also says that \mathcal{E}_ρ is *lower semicontinuous*. Intuitively speaking, inner semicontinuity ensures that the set of optimal payoffs does not suddenly “shrink” as a result of a slight perturbation of the underlying position. This continuity property constitutes the key stability notion in our financial context. Indeed, consider a reference position $X \in \mathcal{X}$ and assume we select an optimal payoff

$$Z \in \mathcal{E}_\rho(X).$$

If the optimal payoff map is inner semicontinuous at X , then for every position $Y \in \mathcal{X}$ the following intuitive robustness property is satisfied:

$$Y \text{ is close to } X \implies \text{there exists a payoff in } \mathcal{E}_\rho(Y) \text{ that is close to } Z.$$

By writing $X = Y + (X - Y)$ we can view the random variable $X - Y$ as a perturbation around X and the above implication tells us that a small perturbation around X should still allow us to find an optimal payoff that is not too different from Z .

For set-valued maps that are convex valued the property of inner semicontinuity is especially powerful in that it allows to ensure the existence of continuous selections, see Theorem 5.58 in Rockafellar, Wets (2009). Here, we say that a function $Z : \mathcal{X} \rightarrow \mathcal{M}$ is a *continuous selection* of \mathcal{E}_ρ if it satisfies

$$Z(X) \in \mathcal{E}_\rho(X)$$

for every $X \in \mathcal{X}$ such that $\mathcal{E}_\rho(X)$ is nonempty and if it is continuous. In other words, a continuous selection for the optimal payoff map can be interpreted as a procedure to select optimal payoffs in a robust way, i.e. so that

$$Y \text{ is close to } X \implies Z(Y) \in \mathcal{E}_\rho(Y) \text{ is close to } Z(X) \in \mathcal{E}_\rho(X).$$

Outer semicontinuity

In this short section we show that the optimal payoff map is always outer semicontinuous on the whole of \mathcal{X} .

Theorem 5.1. *The optimal payoff map \mathcal{E}_ρ is outer semicontinuous at every $X \in \mathcal{X}$.*

Proof. Take an arbitrary $X \in \mathcal{X}$ and fix $Z \in \mathcal{X} \setminus \mathcal{E}_\rho(X)$. Assume first that $Z \notin \mathcal{M}$. In this case, since \mathcal{M} is closed, we find a neighborhood \mathcal{U}_Z of Z satisfying $\mathcal{U}_Z \cap \mathcal{M} = \emptyset$, which implies $\mathcal{E}_\rho(Y) \cap \mathcal{U}_Z = \emptyset$ for every $Y \in \mathcal{X}$. This shows that \mathcal{E}_ρ is outer semicontinuous at X .

Assume now that $Z \in \mathcal{M}$ but $X + Z \notin \mathcal{A}$. Since \mathcal{A} is closed, there exist neighborhoods \mathcal{U}_X of X and \mathcal{U}_Z of Z such that $(\mathcal{U}_X + \mathcal{U}_Z) \cap \mathcal{A} = \emptyset$. In particular, we must have $\mathcal{E}_\rho(Y) \cap \mathcal{U}_Z = \emptyset$ for every $Y \in \mathcal{U}_X$, showing that \mathcal{E}_ρ is outer semicontinuous at X .

Finally, assume that $Z \in \mathcal{M}$ and $X + Z \in \mathcal{A}$ but $\pi(Z) \neq \rho(X)$. In this case, set

$$\varepsilon = \frac{1}{4} |\pi(Z) - \rho(X)|$$

and consider the neighborhoods of X and Z defined, respectively, by

$$\mathcal{U}_X = \{Y \in \mathcal{X}; X - \varepsilon U \leq Y \leq X + \varepsilon U\} \quad \text{and} \quad \mathcal{U}_Z = \{Y \in \mathcal{X}; Z - \varepsilon U \leq Y \leq Z + \varepsilon U\},$$

where U is the strictly positive payoff satisfying assumption (A3). Then, it follows that every element $W \in \mathcal{E}_\rho(Y) \cap \mathcal{U}_Z$ with $Y \in \mathcal{U}_X$ must satisfy

$$\rho(X) - \varepsilon = \rho(X + \varepsilon U) \leq \rho(Y) = \pi(W) = \rho(Y) \leq \rho(X - \varepsilon U) = \rho(X) + \varepsilon$$

as well as

$$\pi(Z) - \varepsilon = \pi(Z - \varepsilon U) \leq \pi(W) \leq \pi(Z + \varepsilon U) = \pi(Z) + \varepsilon.$$

However, this is clearly impossible and we conclude that $\mathcal{E}_\rho(Y) \cap \mathcal{U}_Z = \emptyset$ for every $Y \in \mathcal{U}_X$, proving that \mathcal{E}_ρ is outer semicontinuous at X also in this case. \square

Upper semicontinuity

This section is devoted to investigating necessary and sufficient conditions for the optimal payoff map to be upper semicontinuous. We start by proving a general characterization of upper semicontinuity, which, in line with the explanation provided in the commentary at the end of Chapter 5 in Rockafellar, Wets (2009), highlights the strong link between this stability property and boundedness. In the proof we will make use of some standard notation recalled in the appendix.

Theorem 5.2. *The following statements are equivalent:*

- (a) \mathcal{E}_ρ is upper semicontinuous at every $X \in \mathcal{X}$.
- (b) $\mathcal{E}_\rho(\mathcal{K})$ is compact for every compact set $\mathcal{K} \subset \mathcal{X}$.
- (c) For every $X \in \mathcal{X}$ we have

$$X_n \rightarrow X, Z_n \in \mathcal{E}_\rho(X_n) \implies \exists Z \in \mathcal{E}_\rho(X) : Z_{n_k} \rightarrow Z.$$

Proof. Let us first assume that \mathcal{E}_ρ is upper semicontinuous on \mathcal{X} . If we establish that \mathcal{E}_ρ is compact valued, then it will follow from Lemma 17.8 in Aliprantis, Border (2006) that $\mathcal{E}_\rho(\mathcal{K})$ is compact for every compact set $\mathcal{K} \subset \mathcal{X}$, so that (a) implies (b).

Take an arbitrary $X \in \mathcal{X}$ and recall that $\mathcal{E}_\rho(X)$ is closed. To prove boundedness, assume that $\mathcal{E}_\rho(X) \neq \emptyset$ and consider the open neighborhood of $\mathcal{E}_\rho(X)$ defined by

$$\mathcal{U} = \bigcup_{Z \in \mathcal{E}_\rho(X) \cap \mathcal{B}_1(0)} \text{int } \mathcal{B}_1(Z) \cup \bigcup_{r > 1} \bigcup_{Z \in \mathcal{E}_\rho(X) \cap \text{bd } \mathcal{B}_r(0)} \text{int } \mathcal{B}_{\frac{1}{r}}(Z).$$

Moreover, consider the strictly-increasing function $f : (1, \infty) \rightarrow (2, \infty)$ given by $f(r) = r + \frac{1}{r}$. Note that for every $Y \in \mathcal{U}$ with $\|Y\| > 2$ we find suitable $r > 1$ and $Z \in \mathcal{E}_\rho(X) \cap \text{bd } \mathcal{B}_r(0)$ such that $Y \in \text{int } \mathcal{B}_{\frac{1}{r}}(Z)$, which implies

$$\|Y\| \leq \|Z\| + \|Y - Z\| \leq f(r).$$

As a result, it follows that

$$Y \in \mathcal{U} \setminus \mathcal{B}_2(0) \implies d(Y, \mathcal{E}_\rho(X)) \leq \frac{1}{f^{-1}(\|Y\|)}. \quad (3)$$

By upper semicontinuity we find a radius $r > 0$ such that $\mathcal{E}_\rho(\mathcal{B}_r(X)) \subset \mathcal{U}$. In particular, there exists $\varepsilon > 0$ small enough so that $X + \varepsilon U \in \mathcal{B}_r(X)$ and thus

$$\mathcal{E}_\rho(X) - \varepsilon U = \mathcal{E}_\rho(X + \varepsilon U) \subset \mathcal{U}. \quad (4)$$

We claim that

$$d(\mathcal{E}_\rho(X) - \varepsilon U, \mathcal{E}_\rho(X)) > 0. \quad (5)$$

To see this, take any $Z, W \in \mathcal{E}_\rho(X)$ and note that, since $\pi(Z) = \rho(X) = \pi(W)$, we have $Z - W \in \ker(\pi)$. This yields

$$\|Z - \varepsilon U - W\| \geq \varepsilon d(U, \ker(\pi)) > 0,$$

establishing (5). By combining (3), (4) and (5) we conclude that $\mathcal{E}_\rho(X)$ must be bounded. Indeed, if this were not the case, we could find by (4) an unbounded sequence $(Z_n) \subset \mathcal{E}_\rho(X) - \varepsilon U$ satisfying

$$d(Z_n, \mathcal{E}_\rho(X)) \leq \frac{1}{f^{-1}(\|Z_n\|)} \rightarrow 0$$

by (3). However, this would contradict (5).

Assume now that (b) holds and consider a sequence $(X_n) \subset \mathcal{X}$ and $X \in \mathcal{X}$ such that $X_n \rightarrow X$. Moreover, take $Z_n \in \mathcal{E}_\rho(X_n)$ for every $n \in \mathbb{N}$. Since we can assume without loss of generality that (X_n) is contained in a compact set, it follows that (Z_n) is also contained in a compact set and, hence, admits a convergent subsequence (Z_{n_k}) . Let $Z \in \mathcal{M}$ be the corresponding limit. Since $X_{n_k} + Z_{n_k} \in \mathcal{A}$ for all $k \in \mathbb{N}$, it is immediate to see that $X + Z \in \mathcal{A}$. Moreover, we have

$$\pi(Z) = \lim_{k \rightarrow \infty} \pi(Z_{n_k}) = \lim_{k \rightarrow \infty} \rho(X_{n_k}) = \rho(X)$$

by the continuity of ρ . This shows that $Z \in \mathcal{E}_\rho(X)$ and establishes that (b) implies (c).

We conclude the proof by noting that (c) always implies (a) due to Theorem 17.20 in Aliprantis, Border (2006). \square

Our next result provides an additional sufficient condition for upper semicontinuity in terms of asymptotic cones. In the presence of star shapedness, the condition becomes also necessary. In this case, upper semicontinuity is equivalent to being compact valued.

Theorem 5.3. *Consider the following statements:*

- (i) \mathcal{E}_ρ is upper semicontinuous at every $X \in \mathcal{X}$.
- (ii) $\mathcal{E}_\rho(X)$ is compact for every $X \in \mathcal{X}$.
- (iii) $\mathcal{A}^\infty \cap \ker(\pi) = \{0\}$.

Then, (i) implies (ii) and is implied by (iii). If \mathcal{A} is star shaped and $\mathcal{E}_\rho(X) \neq \emptyset$ for every $X \in \mathcal{X}$, the above statements are all equivalent.

Proof. It follows immediately from Theorem 5.2 that (i) implies (ii). To prove that (iii) implies (i), assume that $\mathcal{A}^\infty \cap \ker(\pi) = \{0\}$ and consider a compact set $\mathcal{K} \subset \mathcal{X}$. We know from Proposition 3.5 that $\mathcal{E}_\rho(\mathcal{K})$ is closed. Hence, if $\mathcal{E}_\rho(\mathcal{K})$ is not compact, we must find a sequence $(X_n) \subset \mathcal{K}$ converging to some $X \in \mathcal{K}$ and elements $Z_n \in \mathcal{E}_\rho(X_n)$ for $n \in \mathbb{N}$ such that $\|Z_n\| \geq n$ for every $n \in \mathbb{N}$. Since the sequence $(\frac{1}{\|Z_n\|}(X_n + Z_n))$ is bounded, we can assume without loss of generality that

$$\frac{1}{\|Z_n\|}(X_n + Z_n) \rightarrow Z$$

for a suitable nonzero payoff $Z \in \mathcal{M}$. Note that $X_n + Z_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ yields $Z \in \mathcal{A}^\infty$. Moreover, Z satisfies

$$\pi(Z) = \lim_{n \rightarrow \infty} \frac{\pi(Z_n)}{\|Z_n\|} = \lim_{n \rightarrow \infty} \frac{\rho(X_n)}{\|Z_n\|} = 0,$$

where we used that $\rho(X_n) \rightarrow \rho(X)$ by the continuity of ρ . This shows that $Z \in \ker(\pi)$. Since this is in contrast with our assumption (ii), we conclude that $\mathcal{E}_\rho(\mathcal{K})$ must be compact and \mathcal{E}_ρ is therefore upper semicontinuous due to Theorem 5.2. This establishes that (iii) implies (i).

Assume now that \mathcal{A} is star shaped. To show the equivalence we only need to prove that (ii) implies (iii). To this effect, assume that \mathcal{E}_ρ is compact valued so that $\mathcal{E}_\rho(X)^\infty = \{0\}$ for every $X \in \mathcal{X}$ by Theorem 3.5 in Rockafellar, Wets (2009). As a result, it follows from Proposition 3.5 that $\mathcal{A}^\infty \cap \ker(\pi) = \{0\}$ holds, showing that (iii) implies (ii) and concluding the proof. \square

We conclude this section by showing that upper semicontinuity always holds whenever the acceptance set is strictly convex. This is a direct consequence of the preceding result once we recall that, by Corollary 4.12, every position admits at most one optimal payoff under a strictly-convex acceptance set.

Corollary 5.4. *Assume \mathcal{A} is strictly convex. Then, \mathcal{E}_ρ is upper semicontinuous at every $X \in \mathcal{X}$.*

Inner semicontinuity

In this section we focus on the (semi)continuity notion that, as discussed above, is most relevant in our framework, namely inner semicontinuity. We start by recalling that inner semicontinuity can be always characterized by means of sequences. Moreover, we show that global inner semicontinuity is automatically ensured once it holds at every point belonging to the intersection between the boundary of \mathcal{A} and the boundary of the “augmented” acceptance set $\mathcal{A} + \ker(\pi)$, which coincides with the set where ρ is zero.

Theorem 5.5. *The following statements are equivalent:*

- (a) \mathcal{E}_ρ is inner semicontinuous at every $X \in \mathcal{X}$.
- (b) \mathcal{E}_ρ is inner semicontinuous at every $X \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$.
- (c) For every $X \in \mathcal{X}$ we have

$$X_n \rightarrow X, Z \in \mathcal{E}_\rho(X) \implies \exists Z_n \in \mathcal{E}_\rho(X_n) : Z_n \rightarrow Z.$$

Proof. It follows from Theorem 17.21 in Aliprantis, Border (2006) that (a) and (c) are always equivalent. Hence, it remains to prove that (b) implies (a). To this effect, assume that \mathcal{E}_ρ is inner semicontinuous on $\text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$ and take any $X \in \mathcal{X}$ and a sequence $(X_n) \subset \mathcal{X}$ such that $X_n \rightarrow X$. Moreover, take an arbitrary $Z \in \mathcal{E}_\rho(X)$. Since $X + Z \in \text{bd } \mathcal{A} \cap \text{bd}(\mathcal{A} + \ker(\pi))$, it follows that \mathcal{E}_ρ is inner semicontinuous at $X + Z$. As a result, noting that $X_n + Z \rightarrow X + Z$ and $0 \in \mathcal{E}_\rho(X_n + Z)$ for every $n \in \mathbb{N}$, we find a suitable sequence $(W_n) \subset \mathcal{M}$ satisfying $W_n \rightarrow 0$ and $W_n \in \mathcal{E}_\rho(X_n + Z)$ for every $n \in \mathbb{N}$. It is now clear that $Z + W_n \rightarrow Z$ and $Z + W_n \in \mathcal{E}_\rho(X_n)$ for all $n \in \mathbb{N}$, proving that \mathcal{E}_ρ is inner semicontinuous at X . \square

Remark 5.6. It is not difficult to verify that point (c) in the above theorem is equivalent to

$$X_n \rightarrow X, Z \in \mathcal{E}_\rho(X) \implies \exists Z_{n_k} \in \mathcal{E}_\rho(X_{n_k}) : Z_{n_k} \rightarrow Z$$

for every position $X \in \mathcal{X}$. This is the equivalent condition for inner semicontinuity provided in Theorem 17.21 in Aliprantis, Border (2006). \square

Inner semicontinuity and polyhedral acceptance sets

In this section we focus on polyhedral acceptance sets and establish the main result of the paper, showing that the optimal payoff map is always inner semicontinuous in the polyhedral case.

First of all, recall from Proposition A.2 that any polyhedral set $\mathcal{A} \subset \mathcal{X}$ can be expressed as

$$\mathcal{A} = \bigcap_{i=1}^m \{X \in \mathcal{X}; \psi_i(X) \geq \sigma_{\mathcal{A}}(\psi_i)\}$$

for suitable functionals $\psi_1, \dots, \psi_m \in \text{bar } \mathcal{A}$. Here, we have denoted by $\sigma_{\mathcal{A}}$ the (lower) support function of \mathcal{A} and by $\text{bar } \mathcal{A}$ the corresponding effective domain, i.e. the so-called barrier cone of \mathcal{A} . Moreover, we can and will always choose ψ_1, \dots, ψ_m in such a way that none of them belongs to the conical hull of the others. In addition, recall that $\text{bar } \mathcal{A} \subset \mathcal{X}'_+$ whenever \mathcal{A} is an acceptance set.

Before we prove inner semicontinuity, it is useful to single out the following simple lemma (we provide a proof since we were unable to find an explicit reference). Here, we denote by $\text{ri } \mathcal{C}$ the *relative interior* of a set $\mathcal{C} \subset \mathcal{X}$. Recall that, if \mathcal{C} is convex, then $\text{ri } \mathcal{C}$ is the nonempty collection of points $X \in \mathcal{C}$ such that for every Y in the conical hull of \mathcal{C} there exists $\varepsilon > 0$ satisfying $X + \varepsilon Y \in \mathcal{C}$ as well as $X - \varepsilon Y \in \mathcal{C}$.

Lemma 5.7. *Assume \mathcal{A} is polyhedral and is represented by $\psi_1, \dots, \psi_m \in \mathcal{X}'_+$. Then, for every nonempty convex subset $\mathcal{C} \subset \text{bd } \mathcal{A}$ the following statements hold:*

- (i) *If $X, Y \in \text{ri } \mathcal{C}$, then $I_a(X) = I_a(Y)$.*
- (ii) *If $X \in \text{ri } \mathcal{C}$ and $Y \in \mathcal{C}$, then $I_a(X) \subset I_a(Y)$.*

Proof. Take any $X \in \text{ri } \mathcal{C}$ and $Y \in \mathcal{C}$. Then, there exists $\varepsilon > 0$ such that we both have $X + \varepsilon(Y - X) \in \mathcal{C}$ and $X - \varepsilon(Y - X) \in \mathcal{C}$. If $i \in I_a(X)$, then it is easy to see that

$$\min\{\sigma_{\mathcal{A}}(\psi_i) + \varepsilon\psi_i(Y - X), \sigma_{\mathcal{A}}(\psi_i) - \varepsilon\psi_i(Y - X)\} \geq \sigma_{\mathcal{A}}(\psi_i),$$

which is only possible if $\psi_i(Y) = \psi_i(X) = \sigma_{\mathcal{A}}(\psi_i)$, i.e. if $i \in I_a(Y)$. Hence, $I_a(X) \subset I_a(Y)$ and the second statement is proved. To establish the first one, it suffices to exchange the roles of X and Y . \square

The next lemma will constitute the key step in the proof of the inner semicontinuity of the optimal payoff map in the presence of polyhedral acceptance sets.

Lemma 5.8. *Assume that \mathcal{A} is polyhedral. Then, there exists a set-valued map $\mathcal{C} : \mathcal{X} \rightrightarrows \mathcal{M}$ such that $\mathcal{C}(X) \neq \emptyset$ and*

$$\mathcal{E}_{\rho}(X) = (\mathcal{A}^{\infty} \cap \ker(\pi)) + \mathcal{C}(X)$$

for every $X \in \mathcal{X}$ and such that $\mathcal{C}(\mathcal{K})$ is bounded for every compact set $\mathcal{K} \subset \mathcal{X}$.

Proof. Let $X \in \mathcal{X}$ be fixed and recall from Proposition 3.5 that, since \mathcal{A} is polyhedral, $\mathcal{E}_{\rho}(X)$ is also polyhedral. We denote by $\mathcal{C}(X)$ the convex hull of the set of the extreme points of the polyhedral set $\mathcal{E}_{\rho}(X) \cap \mathcal{N}$, where \mathcal{N} is any vector space in \mathcal{X} whose intersection with the lineality space of $\mathcal{E}_{\rho}(X)$ is reduced to zero. Note that \mathcal{N} does not depend on the choice of X by virtue of Proposition 3.5. Moreover, note that, by construction, $\mathcal{E}_{\rho}(X) \cap \mathcal{N}$ does not contain any vector space and, thus, does admit extreme points. Now, it follows from Proposition A.1 and from Proposition 3.5 that $\mathcal{E}_{\rho}(X)$ can be decomposed as

$$\mathcal{E}_{\rho}(X) = \mathcal{E}_{\rho}(X)^{\infty} + \mathcal{C}(X) = (\mathcal{A}^{\infty} \cap \ker(\pi)) + \mathcal{C}(X).$$

It remains to prove that \mathcal{C} maps compact sets into bounded sets. To this effect, assume that \mathcal{A} is represented by $\psi_1, \dots, \psi_m \in \mathcal{X}'_+$ and define for every $i \in \{1, \dots, m+2\}$ the maps $\alpha_i : \mathcal{N} \rightarrow \mathbb{R}$ and $\beta_i : \mathcal{X} \rightarrow \mathbb{R}$ by setting

$$\alpha_i(Z) = \begin{cases} \psi_i(Z) & \text{if } i \in \{1, \dots, m\}, \\ \pi(Z) & \text{if } i = m+1, \\ -\pi(Z) & \text{if } i = m+2 \end{cases} \quad \text{and} \quad \beta_i(X) = \begin{cases} \sigma_{\mathcal{A}}(\psi) - \psi_i(X) & \text{if } i \in \{1, \dots, m\}, \\ \rho(X) & \text{if } i = m+1, \\ -\rho(X) & \text{if } i = m+2. \end{cases}$$

Then, it follows from (1) in Proposition 3.5 that the polyhedral set $\mathcal{E}_\rho(X) \cap \mathcal{N}$ can be expressed as

$$\mathcal{E}_\rho(X) \cap \mathcal{N} = \{Z \in \mathcal{N}; \alpha_i(Z) \geq \beta_i(X), \forall i \in \{1, \dots, m+2\}\}$$

for every $X \in \mathcal{X}$. Now, denote by \mathcal{I} the collection of all subsets $I \subset \{1, \dots, m+2\}$ consisting of $d = \dim \mathcal{N}$ elements and such that $\{\alpha_i; i \in I\}$ is linearly independent. The collection \mathcal{I} is nonempty by virtue of Proposition 3.3.3 in Bertsekas, Nedić, Ozdaglar (2003). Moreover, define $\alpha_I : \mathcal{N} \rightarrow \mathbb{R}^d$ and $\beta_I : \mathcal{X} \rightarrow \mathbb{R}^d$ by setting

$$\alpha_I(Z) = (\alpha_{i_1}(Z), \dots, \alpha_{i_d}(Z)) \quad \text{and} \quad \beta_I(X) = (\beta_{i_1}(X), \dots, \beta_{i_d}(X))$$

for every $I = \{i_1, \dots, i_d\} \in \mathcal{I}$ and note that α_I is clearly linear and bijective and β_I is continuous due to the continuity of ρ for every $I \in \mathcal{I}$. As a result of the above-mentioned Proposition 3.3.3, every extreme point of $\mathcal{E}_\rho(X) \cap \mathcal{N}$, with $X \in \mathcal{X}$, has the form $\alpha_I^{-1}(\beta_I(X))$ for some $I \in \mathcal{I}$. This implies that, for any compact set $\mathcal{K} \subset \mathcal{X}$, we have

$$\mathcal{C}(\mathcal{K}) \subset \text{co} \left(\bigcup_{I \in \mathcal{I}} \alpha_I^{-1}(\beta_I(\mathcal{K})) \right).$$

Since \mathcal{I} contains finitely many members and the convex hull of a compact set is still compact, we conclude that the right-hand side above is compact. This establishes that $\mathcal{C}(\mathcal{K})$ is bounded and concludes the proof. \square

We are now ready to prove the announced result establishing the inner semicontinuity of the optimal payoff map in the presence of polyhedral acceptance sets.

Theorem 5.9. *Assume that \mathcal{A} is polyhedral. Then, \mathcal{E}_ρ is inner semicontinuous at every $X \in \mathcal{X}$.*

Proof. We assume throughout that \mathcal{A} is represented by $\psi_1, \dots, \psi_m \in \mathcal{X}'_+$. Now, take $X \in \mathcal{X}$ and consider an arbitrary sequence $(X_n) \subset \mathcal{X}$ converging to X and an arbitrary $Z \in \mathcal{E}_\rho(X)$. Moreover, let $(Z_n) \subset \mathcal{M}$ be any sequence satisfying $Z_n \in \mathcal{C}(X_n)$ for every $n \in \mathbb{N}$, where $\mathcal{C} : \mathcal{X} \rightrightarrows \mathcal{M}$ is the set-valued map from Lemma 5.8. Since (Z_n) is bounded as a consequence of the same result, we can assume without loss of generality that $Z_n \rightarrow W$ for a suitable $W \in \mathcal{M}$. In particular, note that $X_n + Z_n \rightarrow X + W$ so that $X + W \in \mathcal{A}$ and

$$\pi(W) = \lim_{n \rightarrow \infty} \pi(Z_n) = \lim_{n \rightarrow \infty} \rho(X_n) = \rho(X)$$

by the continuity of ρ . In other words, we have $W \in \mathcal{E}_\rho(X)$.

If $|\mathcal{E}_\rho(X)| = 1$, then we must have $W = Z$ and we immediately conclude that \mathcal{E}_ρ is inner semicontinuous at X by virtue of Theorem 5.5. Hence, let us assume that $|\mathcal{E}_\rho(X)| > 1$. Note that, being convex, $\mathcal{E}_\rho(X)$ has a nonempty relative interior in this case.

We first assume that $Z \in \text{ri} \mathcal{E}_\rho(X)$. Since $X + Z \in \text{ri}(X + \mathcal{E}_\rho(X))$ and $X + W \in X + \mathcal{E}_\rho(X)$ and since $X + \mathcal{E}_\rho(X) \subset \text{bd} \mathcal{A}$ by Proposition 3.4, we infer from Lemma 5.7 that

$$I_a(X + Z) \subset I_a(X + W).$$

In particular, it is easy to see that

$$I_a(X + Z) \subset \{i \in \{1, \dots, m\}; \psi_i(Z - W) = 0\}.$$

For $i \in I_a(X + Z)$ we can use the above inclusion to obtain

$$\psi_i(X_n + Z_n + Z - W) = \psi_i(X_n + Z_n) + \psi_i(Z - W) = \psi_i(X_n + Z_n) \geq \sigma_{\mathcal{A}}(\psi_i)$$

for every $n \in \mathbb{N}$. For $i \notin I_a(X + Z)$ we immediately see that

$$\psi_i(X_n + Z_n + Z - W) = \psi_i(X_n + Z_n - X - W) + \psi_i(X + Z) > \sigma_{\mathcal{A}}(\psi_i)$$

for n large enough since $\psi_i(X_n + Z_n - X - W) \rightarrow 0$. By combining the above findings it follows that

$$X_n + Z_n + Z - W \in \mathcal{A}$$

for n large enough. Now, setting for every $n \in \mathbb{N}$

$$W_n = Z_n + Z - W$$

we obtain a sequence in \mathcal{M} converging to Z and eventually satisfying $X_n + W_n \in \mathcal{A}$ and

$$\pi(W_n) = \rho(X_n) + \rho(X) - \rho(X) = \rho(X_n).$$

In other words, we have exhibited a sequence $(W_n) \subset \mathcal{M}$ such that $W_n \in \mathcal{E}_\rho(X_n)$ for n large enough and $W_n \rightarrow Z$. This shows that \mathcal{E}_ρ is inner semicontinuous at X by Theorem 5.5.

The above construction works whenever Z belongs to the relative interior of $\mathcal{E}_\rho(X)$. If Z lies outside the relative interior of $\mathcal{E}_\rho(X)$, then we may approximate it by elements in the relative interior and apply the above construction. We can now apply Theorem 5.5 to conclude that \mathcal{E}_ρ is inner semicontinuous at X . \square

An immediate application of Michael's Selection Theorem, see Theorem 5.58 in Rockafellar, Wets (2009), establishes that \mathcal{E}_ρ always admits a continuous selection in the presence of a polyhedral acceptance set.

Corollary 5.10. *Assume that \mathcal{A} is polyhedral. Then, \mathcal{E}_ρ admits a continuous selection.*

Inner semicontinuity and strictly-convex acceptance sets

We continue our investigation of inner semicontinuity by highlighting that, if the chosen acceptance set is strictly convex, the optimal payoff map is always inner semicontinuous.

Theorem 5.11. *Assume \mathcal{A} is strictly convex. Then, \mathcal{E}_ρ is inner semicontinuous.*

Proof. Take a position $X \in \mathcal{X}$. If $\mathcal{E}_\rho(X) = \emptyset$, then \mathcal{E}_ρ is clearly inner semicontinuous at X . Otherwise, we must have $|\mathcal{E}_\rho(X)| = 1$. Indeed, should we find distinct Z and W in $\mathcal{E}_\rho(X)$, we would have $X + Z \in \text{bd } \mathcal{A}$ as well as $X + W \in \text{bd } \mathcal{A}$ so that

$$X + \lambda Z + (1 - \lambda)W \in \text{int } \mathcal{A}$$

for any given $\lambda \in (0, 1)$ by strict convexity. However, this would allow us to choose $\varepsilon > 0$ small enough to ensure

$$X + \lambda Z + (1 - \lambda)W - \varepsilon U \in \mathcal{A},$$

leading to the contradiction

$$\rho(X) \leq \pi(\lambda Z + (1 - \lambda)W - \varepsilon U) = \lambda \rho(X) + (1 - \lambda) \rho(W) - \varepsilon = \rho(X) - \varepsilon.$$

It is now immediate to verify that, since $|\mathcal{E}_\rho(X)| = 1$ holds, \mathcal{E}_ρ is inner semicontinuous at X , concluding the proof. \square

Inner semicontinuity may fail with nonconvex acceptance sets

It is hardly surprising that inner semicontinuity generally fails in the presence of a nonconvex acceptance set. Here, we discuss the case of acceptability based on VaR and show that, in this case, an arbitrarily small perturbation of a capital position may shrink the set of optimal payoffs from an infinite set to a singleton!

Example 5.12. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and define a probability measure \mathbb{P} on the power set of Ω by setting $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = \alpha$ and $\mathbb{P}(\{\omega_3\}) = 1 - 2\alpha$ for a given $\alpha \in (0, \frac{1}{2})$. In this framework, consider the nonconvex acceptance set

$$\mathcal{A}_{\text{VaR}}(\alpha) = \{X \in \mathcal{X}; \mathbb{P}(X < 0) \leq \alpha\} = \{X \in \mathcal{X}; \max\{X(\omega_1), X(\omega_2)\} \geq 0, X(\omega_3) \geq 0\}.$$

Moreover, consider the random variables $U = 1_\Omega$ and $Z = 1_{\{\omega_1\}} - 1_{\{\omega_3\}}$ and let \mathcal{M} be the vector space spanned by them. The corresponding pricing functional is specified by setting $\pi(U) = 1$ and $\pi(Z) = 0$. Note that, under these specifications, our initial assumptions (A1) to (A4) are all satisfied. Now, it is easy to verify that

$$\mathcal{E}_\rho(0) = \{aZ; a \in (-\infty, 0]\}.$$

However, for every $n \in \mathbb{N}$ the position $X_n = -\frac{1}{n}1_{\{\omega_2\}}$ satisfies

$$\rho(X_n) = \inf_{a \in \mathbb{R}} \max \{a, \min \{\frac{1}{n}, -a\}\} = 0$$

so that one obtains $\mathcal{E}_\rho(X_n) = \{0\}$. This shows that \mathcal{E}_ρ fails to be inner semicontinuous at zero. \square

Inner semicontinuity may fail with convex acceptance sets

The failure of inner semicontinuity we have illustrated above in the context of acceptability based on VaR critically depends upon the nonconvexity of the acceptance set. Since convexity is often associated with continuity properties, one could expect that we might gain inner semicontinuity by moving to a convex environment. However, this is unfortunately not true and, in fact, the same negative result obtained in the context of VaR acceptability can be reproduced with a convex acceptance set. More precisely, we construct a convex acceptance set \mathcal{A} and exhibit a sequence (X_n) converging to a capital position X such that

$$|\mathcal{E}_\rho(X_n)| = 1$$

for every $n \in \mathbb{N}$ and yet

$$|\mathcal{E}_\rho(X)| = \infty.$$

This implies that \mathcal{E}_ρ cannot be inner semicontinuous at X and shows that, even though the acceptance set \mathcal{A} is convex, a slight perturbation of X may drastically reduce the range of optimal payoffs.

We assume that $|\Omega| = 3$ and, for the sake of readability, identify a random variable $X \in \mathcal{X}$ with the vector

$$x = (x_1, x_2, x_3) = (X(\omega_1), X(\omega_2), X(\omega_3)) \in \mathbb{R}^3.$$

The acceptance set will be obtained by a suitable rotation applied to the following basic set.

1. The basic set. Fix $r > 0$ and define a subset of \mathbb{R}^3 by setting

$$\mathcal{B}_r = \left\{ x \in \mathbb{R}^2 \times (0, 1]; \frac{x_1^2}{1 + r^2 x_3} + \frac{x_2^2}{r^2 x_3} \leq 1 \right\} \cup \{x \in \mathbb{R}^3; x_1 \in [-1, 1], x_2 = x_3 = 0\}.$$

The lower boundary of this set has a boat-like shape depicted in Figure 1, with the projection of some level sets. For every $h \in [0, 1]$ the slice $\mathcal{S}(h) = \{x \in \mathcal{B}_r; x_3 = h\}$ is an ellipsoid centered in 0 with axes parallel to the canonical axes e_1 and e_2 and of lengths $2\sqrt{1 + r^2 h}$ and $2r\sqrt{h}$, respectively. In particular, the slice $\mathcal{S}(0)$ is a degenerated ellipsoid. Moreover, observe that the slice $\mathcal{S}(1)$ contains a circle of radius r .

The set \mathcal{B}_r is clearly closed and is easily seen to be convex. Indeed, since the function $g : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ defined by $g(s, t) = s^2/t$ is convex, it follows that $\lambda x + (1 - \lambda)y \in \mathcal{B}_r$ for every $x, y \in \mathcal{B}_r$ with $x_3 > 0$ and $y_3 > 0$ and $\lambda \in [0, 1]$. The same conclusion holds for every $x, y \in \mathcal{B}_r$ by closedness.

For every given radius $0 < R \leq \frac{r^2}{2\sqrt{1+r^2}}$ consider the ice-cream cone

$$\mathcal{K}_R = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 \leq R^2 x_3\}.$$

We claim that \mathcal{B}_r satisfies

$$(\mathcal{B}_r + \mathcal{K}_R) \cap \{x \in \mathbb{R}^3; x_3 \leq 1\} = \mathcal{B}_r. \quad (6)$$

To prove this, consider the convex function $f_r : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f_r(x_1, x_2) = \frac{x_1^2 + x_2^2 - 1 + \sqrt{(x_1^2 + x_2^2 - 1)^2 + 4x_2^2}}{2r^2}. \quad (7)$$

After some elementary manipulations, one can show that \mathcal{B}_r is nothing but a section of the epigraph of f_r , namely

$$\mathcal{B}_r = \{x \in \mathbb{R}^2 \times [0, 1]; f_r(x_1, x_2) \leq x_3\}.$$

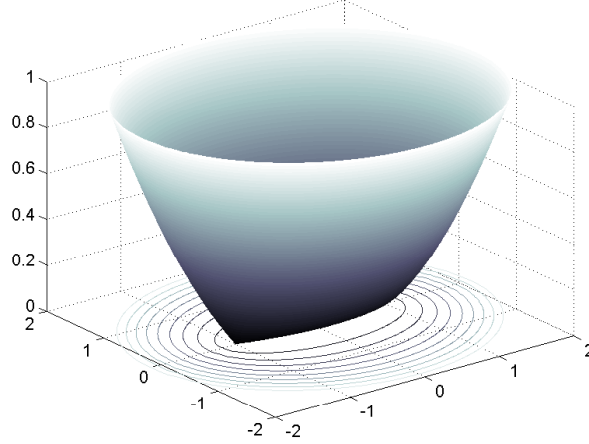


Figure 1: The set \mathcal{B}_r for $r = 2$.

For every $x \in \text{bd } \mathcal{B}_r$ with $x_3 \in (0, 1)$ one can easily verify that

$$\frac{\partial f_r(x_1, x_2)}{\partial x_1} = \frac{1}{2r^2} \frac{2x_1 f_r(x_1, x_2)}{f_r(x_1, x_2) - (x_1^2 + x_2^2 - 1)} \quad \text{and} \quad \frac{\partial f_r(x_1, x_2)}{\partial x_2} = \frac{1}{2r^2} \frac{2x_2 (f_r(x_1, x_2) + 2)}{f_r(x_1, x_2) - (x_1^2 + x_2^2 - 1)}.$$

Since $f_r(x_1, x_2) = x_3$ and $x_1^2 + x_2^2 - 1 = r^2 x_3 - x_2^2 / (r^2 x_3)$, we infer that

$$\begin{aligned} \|\nabla f_r(x_1, x_2)\|_2^2 &= \frac{1}{4r^4} \frac{4(x_1^2 + x_2^2) f_r^2(x_1, x_2) + 16x_2^2 f_r(x_1, x_2) + 16x_2^2}{(f_r(x_1, x_2) - (x_1^2 + x_2^2 - 1))^2} \\ &= \frac{1}{4r^4} \frac{16r^4 x_3^2 (1 + r^2 x_3)}{r^4 x_3^2 + x_2^2} \\ &\leq \frac{4(1 + r^2)}{r^4}. \end{aligned}$$

It follows that (6) is satisfied. To see this, take any $x \in \mathcal{B}_r$ with $x_3 \in (0, 1)$ and $y \in \mathcal{K}_R$ such that $x_3 + y_3 \leq 1$. The uniform bound on the norm of the gradient established above allows us to write

$$f_r(x_1 + y_1, x_2 + y_2) \leq f_r(x_1, x_2) + \frac{2\sqrt{1+r^2}}{r^2} \sqrt{y_1^2 + y_2^2} \leq x_3 + \frac{2\sqrt{1+r^2}}{r^2} R y_3 \leq x_3 + y_3$$

where we used that $R \leq \frac{r^2}{2\sqrt{1+r^2}}$. By continuity, the inequality $f_r(x_1 + y_1, x_2 + y_2) \leq x_3 + y_3$ can be extended to any $x \in \mathcal{B}_r$ with $x_3 = 0$ and $y \in \mathcal{K}_R$ such that $y_3 \leq 1$. This establishes (6).

2. The rotation. Consider the isometry $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (8)$$

It can be easily verified that Φ is the composition of a clockwise rotation of an angle $\vartheta = \pi/4$ around the unit vector e_3 with a clockwise rotation of an angle ϑ such that $\sin(\vartheta) = \sqrt{2}/\sqrt{3}$ and $\cos(\vartheta) = 1/\sqrt{3}$ around the unit vector $(1/\sqrt{2}, -1/\sqrt{2}, 0)$.

3. The failure of inner semicontinuity. Let $r = 3$ and consider the convex acceptance set $\mathcal{A} \subset \mathbb{R}^3$ defined by

$$\mathcal{A} = \Phi(\mathcal{B}_r) + \mathbb{R}_+^3.$$

Moreover, consider the space of eligible payoffs

$$\mathcal{M} = \Phi(\{x \in \mathbb{R}^3; x_2 = 0\}),$$

which coincides with the vector space spanned by $\Phi(0, 0, \sqrt{3}) = (1, 1, 1)$ and $\Phi(\sqrt{2}, 0, 0) = (1, -1, 0)$, and the pricing functional $\pi : \mathcal{M} \rightarrow \mathbb{R}$ given by

$$\pi(\Phi(x)) = \frac{x_3}{\sqrt{3}}, \quad x \in \Phi^{-1}(\mathcal{M}).$$

In particular, the space $\ker(\pi)$ is spanned by $\Phi(\sqrt{2}, 0, 0) = (1, -1, 0)$. It is easy to see that all our assumptions (A1) to (A4) are satisfied.

Clearly, for every vector $x \in \mathbb{R}^3$ we have

$$\rho(\Phi(x)) = \inf\{\pi(\Phi(z)); z \in \Phi^{-1}(\mathcal{M}), x + z \in \mathcal{B}_3 + \Phi^{-1}(\mathbb{R}_+^3)\}$$

and, similarly,

$$\mathcal{E}_\rho(\Phi(x)) = \Phi(\{z \in \Phi^{-1}(\mathcal{M}); x + z \in \mathcal{B}_3 + \Phi^{-1}(\mathbb{R}_+^3), \rho(\Phi(x)) = \pi(\Phi(z))\}).$$

Set $R = \sqrt{2}$ so that $\Phi^{-1}(\mathbb{R}_+^3) \subset \mathcal{K}_R$. Since $R \leq \frac{r^2}{2\sqrt{1+r^2}}$, it follows from (6) that

$$(\mathcal{B}_r + \Phi^{-1}(\mathbb{R}_+^3)) \cap \{x \in \mathbb{R}^3; x_3 \leq 1\} = \mathcal{B}_r.$$

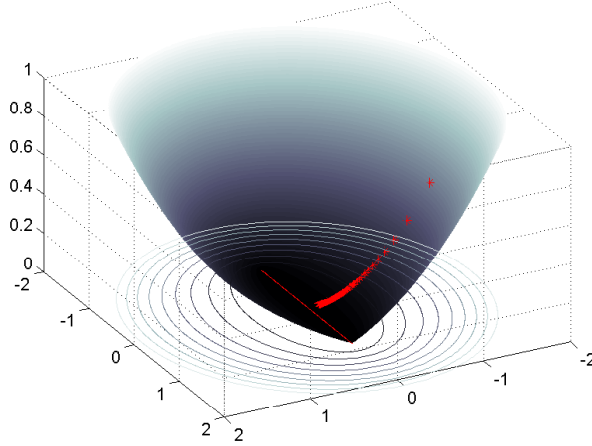


Figure 2: In red: some sets $\Phi^{-1}(\mathcal{E}_\rho(\Phi(x_n)))$ and the segment $\Phi^{-1}(\mathcal{E}_\rho(\Phi(0)))$.

Consider now the sequence in \mathcal{B}_r with general term

$$x_n = (0, r/\sqrt{n}, 1/n).$$

Note that $\Phi(x_n) \rightarrow 0$ and for every $n \in \mathbb{N}$ we have $\rho(\Phi(x_n)) = 0$ and

$$\mathcal{E}_\rho(\Phi(x_n)) = \Phi(\{0\}) = \{0\}.$$

However, it clearly holds

$$\begin{aligned} \mathcal{E}_\rho(\Phi(0)) &= \Phi(\{x \in \mathbb{R}^3; x_1 \in [-1, 1], x_2 = x_3 = 0\}) \\ &= \{\lambda\Phi(-1, 0, 0) + (1 - \lambda)\Phi(1, 0, 0); \lambda \in [0, 1]\}. \end{aligned}$$

This shows that \mathcal{E}_ρ fails to be inner semicontinuous at $\Phi(0) = 0$.

Remark 5.13. Although not inner semicontinuous, the optimal payoff map \mathcal{E}_ρ in the above example is easily seen to admit a continuous selection. Hence, one may wonder whether \mathcal{E}_ρ always admits a continuous selection when the underlying acceptance set is convex. This is, however, not true. Indeed, it is possible to modify the preceding example in such a way that \mathcal{E}_ρ fails to admit a continuous selection even though \mathcal{A} remains convex. Since the example requires some technical work, we relegate its explicit formulation and the corresponding verifications to Appendix B.

6 Conclusions

In this paper we investigated a variety of properties of set-valued maps that arise naturally when studying financial problems such as the adequate capitalization of financial institutions, the pricing and hedging of financial contracts, and the management of systemic risk. In our interpretation of the mathematical results, we focused mainly on a capital regime in which financial institutions are allowed to reach acceptability by raising capital and investing it in a portfolio of pre-specified eligible assets. In this context, these set-valued maps associate to each financial position the set of optimal payoffs that allow an institution to reach acceptability at the lowest cost, i.e. by raising the least amount of capital.

We provided characterizations of the existence and uniqueness of optimal eligible payoffs and focused on the critical property of “stability, i.e. the property that a slight mismeasurement of the initial position does not result in a serious misrepresentation of capital requirements. This requires studying the continuity, and, most importantly, the inner semicontinuity, properties of the above mentioned set-valued maps. As a consequence of inner semicontinuity, one obtains the existence of a continuous selection. In this respect we established that, whenever the acceptance set is polyhedral or strictly convex, we have inner semicontinuity. We put these results in context by providing examples of nonconvex and convex acceptance sets for which inner semicontinuity or the existence of a continuous selection fail. Our results apply to acceptability criteria based on Expected Shortfall, Test Scenarios and Shortfall Risk. On the other hand, our examples show that, if the acceptance set is based on Value-at-Risk, it is not possible to ensure stability as described above.

The main objective of this paper was to provide a comprehensive picture on a number of important finance theoretical questions that, as mentioned in the introduction, do not seem to have been addressed in the literature so far. Even though most of our preliminary results hold in full generality, the key result on the inner semicontinuity under polyhedrality relies explicitly on the underlying finite-dimensional structure. For this reason, we have stated all results in the setting of a finite probability space. A natural further step would be to investigate whether and to which extent the above results can be extended to a more general (infinite-dimensional) framework.

A Notation and terminology

We denote by \mathcal{X} the real vector space of all functions $X : \Omega \rightarrow \mathbb{R}$ where Ω is a given finite set. The space \mathcal{X} becomes a partially ordered normed space when equipped with the pointwise order

$$X \geq Y : \Longleftrightarrow X(\omega) \geq Y(\omega) \text{ for all } \omega \in \Omega$$

and with the maximum norm

$$\|X\| := \max_{\omega \in \Omega} |X(\omega)|.$$

The symbol 0 stands for the null function and the set of positive elements of \mathcal{X} will be denoted by

$$\mathcal{X}_+ := \{X \in \mathcal{X} ; X \geq 0\}.$$

The *interior*, the *closure* and the *boundary* of a set $\mathcal{A} \subset \mathcal{X}$ will be denoted by $\text{int } \mathcal{A}$, $\text{cl } \mathcal{A}$ and $\text{bd } \mathcal{A}$, respectively. Note that \mathcal{X}_+ has nonempty interior. The *closed ball* of radius $r \in (0, \infty)$ centered at $X \in \mathcal{X}$ is denoted by

$$\mathcal{B}_r(X) := \{Y \in \mathcal{X} ; \|Y - X\| \leq r\}.$$

Given two subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$, we denote the *distance* between \mathcal{A} and \mathcal{B} by setting

$$d(\mathcal{A}, \mathcal{B}) := \inf\{\|X - Y\|; X \in \mathcal{A}, Y \in \mathcal{B}\}.$$

If $\mathcal{A} = \{X\}$ for some $X \in \mathcal{X}$ we will simply write $d(X, \mathcal{B})$.

A subset $\mathcal{A} \subset \mathcal{X}$ is said to be *star shaped (about zero)* whenever $X \in \mathcal{A}$ implies that $\lambda X \in \mathcal{A}$ for every $\lambda \in [0, 1]$. We say that \mathcal{A} is *convex* if $\lambda X + (1 - \lambda)Y \in \mathcal{C}$ for every $X, Y \in \mathcal{A}$ and $\lambda \in [0, 1]$ and a *cone* if $\lambda X \in \mathcal{A}$ for every $X \in \mathcal{A}$ and $\lambda \in [0, \infty)$. The smallest convex set containing \mathcal{A} is called the *convex hull* of \mathcal{A} and is denoted by $\text{co } \mathcal{A}$. The smallest convex cone containing \mathcal{A} is called the *conical hull* of \mathcal{A} . Clearly, every convex set containing zero as well as every cone is star shaped.

We say that \mathcal{A} is *polyhedral* if it can be represented as a finite intersection of halfspaces, i.e. if

$$\mathcal{A} = \bigcap_{i=1}^m \{X \in \mathcal{X}; \psi_i(X) \geq \alpha_i\}$$

for suitable linear (hence continuous) functionals $\psi_1, \dots, \psi_m : \mathcal{X} \rightarrow \mathbb{R}$ and scalars $\alpha_1, \dots, \alpha_m \in \mathbb{R}$. Clearly, any polyhedral set is automatically closed and convex.

Recall that the *asymptotic cone* of \mathcal{A} is the closed cone defined by

$$\mathcal{A}^\infty := \bigcap_{\varepsilon > 0} \text{cl}\{\lambda X; \lambda \in [0, \varepsilon], X \in \mathcal{A}\}.$$

Equivalently, \mathcal{A}^∞ consists of all the limits of sequences $(\lambda_n X_n)$ where $(\lambda_n) \subset [0, \infty)$ and $\lambda_n \rightarrow 0$ and $(X_n) \subset \mathcal{A}$. If \mathcal{A} is closed and is either convex or star shaped, the asymptotic cone of \mathcal{A} coincides with the (otherwise smaller) *recession cone* of \mathcal{A} defined by

$$\text{rec } \mathcal{A} := \{X \in \mathcal{X}; Y + \lambda X \in \mathcal{A}, \forall Y \in \mathcal{A}, \forall \lambda \in (0, \infty)\}.$$

Moreover, the *lineality space* of \mathcal{A} is the vector space defined by

$$\text{lin } \mathcal{A} := \mathcal{A}^\infty \cap (-\mathcal{A}^\infty).$$

If \mathcal{A} is convex, then a point $X \in \mathcal{A}$ is said to be an *extreme point* of \mathcal{A} whenever $\mathcal{A} \setminus \{X\}$ is still convex. The set of extreme points of \mathcal{A} is denoted by $\text{ext } \mathcal{A}$. Recall that a convex set admits extreme points if, and only if, it does not contain any vector space. Moreover, any polyhedral set admits at most finitely many extreme points. We refer to Lemma 16.2 and Lemma 16.3 in Barvinok (2002) for a proof of the following useful decomposition of closed convex sets.

Proposition A.1. *Assume that $\mathcal{A} \subset \mathcal{X}$ is closed and convex. Then, there exists a closed convex set $\mathcal{C} \subset \mathcal{X}$ that contains no vector space and satisfies $\text{span } \mathcal{C} \cap \text{lin } \mathcal{A} = \{0\}$ and*

$$\mathcal{A} = \text{lin } \mathcal{A} + \mathcal{C} = \mathcal{A}^\infty + \text{co } \text{ext } \mathcal{C}.$$

We will denote by \mathcal{X}' the topological dual of \mathcal{X} , i.e. the vector space of all linear (hence continuous) functionals $\psi : \mathcal{X} \rightarrow \mathbb{R}$. A linear functional is said to be *positive* if $\psi(X) \geq 0$ whenever $X \in \mathcal{X}_+$. The set of all positive linear functionals is denoted by

$$\mathcal{X}'_+ = \{\psi \in \mathcal{X}'; \psi(X) \geq 0, \forall X \in \mathcal{X}_+\}.$$

The *kernel* of a functional $\psi \in \mathcal{X}'$ is the set defined by

$$\ker(\psi) := \{X \in \mathcal{X}; \psi(X) = 0\}.$$

The set of all linear functionals whose kernel contains \mathcal{A} is called the *annihilator* of \mathcal{A} and is denoted by

$$\mathcal{A}^\perp := \{\psi \in \mathcal{X}'; \psi(X) = 0, \forall X \in \mathcal{A}\}.$$

In addition, the *dual cone* of \mathcal{A} is the set defined by

$$\mathcal{A}^+ := \{\psi \in \mathcal{X}' ; \psi(X) \geq 0, \forall X \in \mathcal{A}\}.$$

The (lower) *support function* of a set $\mathcal{A} \subset \mathcal{X}$ is the map $\sigma_{\mathcal{A}} : \mathcal{X}' \rightarrow \mathbb{R} \cup \{-\infty\}$ given by

$$\sigma_{\mathcal{A}}(\psi) := \inf_{X \in \mathcal{A}} \psi(X).$$

The effective domain of $\sigma_{\mathcal{A}}$, which is easily seen to be a convex cone, is called the *barrier cone* of \mathcal{A} and is denoted by $\text{bar } \mathcal{A}$. We collect in the following proposition a variety of well-known properties of support functions and barrier cones; see e.g. Aubin, Ekeland (2006) or Auslender, Teboulle (2003). We also refer to Schrijver (1998) for the statements about polyhedral sets.

Proposition A.2. *The following statements hold for any subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$:*

- (i) $\sigma_{\mathcal{A}+\mathcal{B}}(\psi) = \sigma_{\mathcal{A}}(\psi) + \sigma_{\mathcal{B}}(\psi)$ for every $\psi \in \mathcal{X}'$.
- (ii) $\text{bar}(\mathcal{A} + \mathcal{B}) = \text{bar } \mathcal{A} \cap \text{bar } \mathcal{B}$.
- (iii) If $\mathcal{A} + \mathcal{X}_+ \subset \mathcal{A}$, then $\text{bar } \mathcal{A} \subset \mathcal{X}'_+$.
- (iv) If \mathcal{A} is a vector space, then $\text{bar } \mathcal{A} = \mathcal{A}^\perp$.
- (v) If \mathcal{A} is a cone, then $\text{bar } \mathcal{A} = \mathcal{A}^+$.
- (vi) If \mathcal{A} is closed and convex, then

$$\mathcal{A} = \bigcap_{\psi \in \text{bar } \mathcal{A}} \{X \in \mathcal{X} ; \psi(X) \geq \sigma_{\mathcal{A}}(\psi)\}.$$

Moreover, for any $X \in \mathcal{X}$ we have $X \in \text{bd } \mathcal{A}$ if and only if $\psi(X) = \sigma_{\mathcal{A}}(\psi)$ for some $\psi \in \text{bar } \mathcal{A}$.

(vii) If \mathcal{A} is polyhedral, then there exist $\psi_1, \dots, \psi_m \in \text{bar } \mathcal{A}$ such that

$$\mathcal{A} = \bigcap_{i=1}^m \{X \in \mathcal{X} ; \psi_i(X) \geq \sigma_{\mathcal{A}}(\psi_i)\}. \quad (9)$$

Moreover, we can always choose ψ_1, \dots, ψ_m in such a way that none of them belongs to the conical hull of the others. In particular, $\text{bar } \mathcal{A}$ is the conical hull of ψ_1, \dots, ψ_m . If ψ_1, \dots, ψ_m satisfy this “minimality” condition, we call (9) the (canonical) representation of \mathcal{A} .

B A continuous selection may not exist

In this section we prove the claim stated in Remark 5.13, namely that the optimal payoff map \mathcal{E}_ρ may fail to admit a continuous selection even though the underlying acceptance set is convex. To this effect, we follow the general construction of the example used to show that \mathcal{E}_ρ may fail to be inner semicontinuous in the presence of a convex acceptance set. The only difference is that we will “twist” the above set \mathcal{B}_r in a suitable way. This modification will require some technical preliminary work to ensure convexity, which was automatic in the above example.

As above, we assume that $|\Omega| = 3$ and, for the sake of readability, identify a random variable $X \in \mathcal{X}$ with the vector

$$x = (x_1, x_2, x_3) = (X(\omega_1), X(\omega_2), X(\omega_3)) \in \mathbb{R}^3.$$

1. A useful set. For any fixed $r > 0$ and for any set $E \subset \mathbb{R}^2$ we write

$$\mathcal{B}_r(E) = \{(u\sqrt{1+r^2x_3}, vr\sqrt{x_3}, x_3) \in \mathbb{R}^3 ; (u, v) \in E, x_3 \in [0, 1]\}.$$

In the sequel we will often consider sets of the form

$$E = \{(u, v) \in \mathbb{R}^2 ; \alpha(u - a)^2 + \beta v^2 \leq 1\}$$

for given $a \in \mathbb{R}$ and $\alpha, \beta > 0$. In this case, the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by setting

$$F(x) = \frac{r^2 x_3}{\beta} (\alpha(u(x_1, x_3) - a)^2 - 1) + x_2^2, \quad (10)$$

where $u(x_1, x_3) = \frac{x_1}{\sqrt{1+r^2 x_3}}$, is easily seen to satisfy

$$\mathcal{B}_r(E) = \{x \in \mathbb{R}^3; F(x) \leq 0, x_3 \in [0, 1]\}.$$

Now, assume that $|a| < \frac{1}{\sqrt{\alpha}}$ so that $0 \in \text{int } E$. The corresponding function F is infinitely differentiable and satisfies

$$\frac{\partial F(x)}{\partial x_3} = \frac{r^2}{\beta} \left((\alpha(u(x_1, x_3) - a)^2 - 1) - \frac{\alpha r^2 x_3 (u(x_1, x_3) - a) u(x_1, x_3)}{1 + r^2 x_3} \right) \neq 0 \quad (11)$$

whenever $x \in \mathcal{B}_r(E)$ with $x_3 > 0$. To see this, assume that $\frac{\partial F(x)}{\partial x_3} = 0$ for some $x \in \mathcal{B}_r(E)$ with $x_3 > 0$. Then, we would have the contradiction

$$\begin{aligned} 0 &\geq \alpha(u(x_1, x_3) - a)^2 - 1 \\ &= r^2 x_3 (1 - \alpha a^2) + \alpha r^2 x_3 u(x_1, x_3) a \\ &\geq r^2 x_3 (1 - \alpha |a(a - u(x_1, x_3))|) \\ &\geq r^2 x_3 (1 - \sqrt{\alpha} |a|) \\ &> 0 \end{aligned}$$

where we used that $|a - u(x_1, x_3)| \leq \frac{1}{\sqrt{\alpha}}$ in the second-to-last inequality and $|a| < \frac{1}{\sqrt{\alpha}}$ in the last inequality. As a result, (11) holds.

As a consequence of the Implicit Function Theorem, there exists an open set $\mathcal{U} \subset \mathbb{R}^2$ and a continuously differentiable function $f : \mathcal{U} \rightarrow (0, \infty)$ for which

$$\mathcal{B}_r(E) \cap \{x \in \mathbb{R}^3; x_3 > 0\} = \{x \in \mathbb{R}^3; f(x_1, x_2) \leq x_3, x_3 \in (0, 1]\}.$$

Of course, it is not difficult to see that we can extend this function continuously to obtain

$$\mathcal{B}_r(E) = \{x \in \mathbb{R}^3; f(x_1, x_2) \leq x_3, x_3 \in [0, 1]\}. \quad (12)$$

2. Convexity of $\mathcal{B}_r(E)$. To establish that $\mathcal{B}_r(E)$ is convex we shall use the following result by Crouzeix [13].

Theorem B.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function. Then, f is convex if and only if the function $\sigma_s : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ defined by*

$$\sigma_s(t) = \sup \left\{ \sum_{i=1}^n s_i x_i; x \in \mathbb{R}^n, f(x) \leq t \right\}$$

is concave for every $s \in \mathbb{R}^n$.

Note first that f is lower semicontinuous and quasiconvex by construction. We use Theorem B.1 to prove its convexity. For every $t \in [0, 1]$ the level set $\mathcal{L}(t) = \{(x_1, x_2) \in \mathbb{R}^2; f(x_1, x_2) \leq t\}$ is easily seen to satisfy

$$\mathcal{L}(t) = \{(u\sqrt{1+r^2 t}, vr\sqrt{t}); \alpha(u-a)^2 + \beta v^2 \leq 1\}.$$

This set is an ellipsoid and its support function can be computed explicitly. For $s \in \mathbb{R}^2$ we indeed have

$$\sigma_s(t) = \sup_{(x_1, x_2) \in \mathcal{L}(t)} s_1 x_1 + s_2 x_2 = \sqrt{\frac{s_1^2(1+r^2 t)}{\alpha} + \frac{s_2^2 r^2 t}{\beta}} + a s_1 \sqrt{1+r^2 t}.$$

As a function of t , we have a sum of two square roots of affine functions. If $as_1 \geq 0$, these two terms are concave. Assume, then, that $as_1 < 0$. The second derivative of the first term is

$$-\frac{1}{4} \left(\frac{s_1^2 r^2}{\alpha} + \frac{s_2^2 r^2}{\beta} \right)^2 \left(\frac{s_1^2(1+r^2t)}{\alpha} + \frac{s_2^2 r^2 t}{\beta} \right)^{-3/2},$$

which can be seen to become more and more negative as s_2^2 increases. So, to establish the concavity of σ_s , it suffices to consider $s_2 = 0$. In this case, we get

$$\sigma_s(t) = \sqrt{\frac{s_1^2(1+r^2t)}{\alpha}} - \sqrt{a^2 s_1^2(1+r^2t)} = |s_1| \sqrt{1+r^2t} \left(\frac{1}{\sqrt{\alpha}} - a \right),$$

which is concave as $a\sqrt{\alpha} < 1$. This proves that f is convex and, hence, that $\mathcal{B}_r(E)$ is also convex.¹

3. Curvature of $\mathcal{B}_r(E)$. For any $x \in \mathbb{R}^3$ with $x_3 > 0$ it follows from the Implicit Function Theorem that

$$\|\nabla f(x_1, x_2)\|_2 = \sqrt{\left(\frac{\partial F(x)}{\partial x_1} \right)^2 + \left(\frac{\partial F(x)}{\partial x_2} \right)^2} \left(\left| \frac{\partial F(x)}{\partial x_3} \right| \right)^{-1}.$$

Now, assume that $a \geq 0$ and $\alpha < 5$ and take $r > \max\{\sqrt{2}, \frac{1}{\sqrt{5-\alpha}}\}$. We claim that

$$\max \left\{ \|\nabla f(x_1, x_2)\|_2; 0 \leq u(x_1, x_3) - a \leq \frac{1}{\sqrt{\alpha}}, x \in \text{bd } \mathcal{B}_r(E) \right\} \leq \frac{2}{r} \quad (13)$$

and, under the assumption $8 > 9\alpha a^2$,

$$\max \left\{ \|\nabla f(x_1, x_2)\|_2; -\frac{1}{\sqrt{\alpha}} \leq u(x_1, x_3) - a \leq \frac{1}{\sqrt{\alpha}}, x \in \text{bd } \mathcal{B}_r(E) \right\} \leq \frac{16 \max\{\frac{\alpha r^2}{r^2+1}, 1\}}{r(8-9\alpha a^2)}. \quad (14)$$

Note that we can restrict the above optimization domains to those $x \in \text{bd } \mathcal{B}_r(E)$ such that $f(x_1, x_2) = x_3 = 1$ by convexity. After a few elementary rearrangements, we get

$$\begin{aligned} \|\nabla f(x_1, x_2)\|_2^2 &= \left(\left(\frac{\partial F(x)}{\partial x_1} \right)^2 + \left(\frac{\partial F(x)}{\partial x_2} \right)^2 \right) \left(\frac{\partial F(x)}{\partial x_3} \right)^{-2} \\ &= \frac{4(1+r^2)}{r^2} \cdot \frac{\alpha^2 r^2 (u(x_1, x_3) - a)^2 + (1 - \alpha(u(x_1, x_3) - a)^2)(1+r^2)}{((1 - \alpha(u(x_1, x_3) - a)^2)(1+r^2) + 3\alpha r^2 (u(x_1, x_3) - a)u(x_1, x_3))^2} \\ &= \frac{4(1+r^2)}{r^2} \cdot \frac{\alpha^2 r^2 (u(x_1, x_3) - a)^2 + (1 - \alpha(u(x_1, x_3) - a)^2)(1+r^2)}{(\alpha(2r^2 - 1)(u(x_1, x_3) - a)^2 + 3\alpha r^2 (u(x_1, x_3) - a) + (1+r^2))^2} \\ &= \frac{4(1+r^2)}{r^2} \phi(t) \end{aligned}$$

where, for notational convenience, we have set

$$\phi(t) = \frac{At^2 + B}{(Ct^2 + Dt + B)^2},$$

with $t = u(x_1, x_3) - a$, $A = \alpha(\alpha r^2 - 1 - r^2)$, $B = 1 + r^2$, $C = \alpha(2r^2 - 1)$, and $D = 3\alpha r^2$. Note that B, C, D are all nonnegative while the sign of A depends on α . We show that $t \geq 0$ implies

$$\phi(t) \leq \phi(0) = \frac{1}{B} = \frac{1}{1+r^2}. \quad (15)$$

¹The same result holds even if $a\sqrt{\alpha} = 1$. To see this, define $E(n) = \{(u, v) \in \mathbb{R}^2; \alpha(u - a_n)^2 + \beta v^2 \leq 1\}$ for any $n \in \mathbb{N}$, where $a_n \rightarrow a$ under the assumption that none of the ellipsoids $E(n)$ is empty. Now, let $x \in \mathcal{B}_r(E)$. Then, there exists a sequence $x_n \rightarrow x$ such that $x_n \in \mathcal{B}_r(E(n))$ for every $n \in \mathbb{N}$. This result is trivial when $x_3 = 0$. Otherwise, setting $v(x_2, x_3) = \frac{x_2}{r\sqrt{x_3}}$ so that $\alpha(u(x_1, x_3) - a)^2 + \beta v(x_2, x_3)^2 \leq 1$, we simply take $u_n(x_1, x_3) = u(x_1, x_3) - a + a_n$, $v_n(x_2, x_3) = v(x_2, x_3)$, and $(x_n)_3 = x_3$, and construct the corresponding point x_n of $E(n)$. Conversely, if $x_n \in \mathcal{B}_r(E(n))$ defines a sequence converging to x with $x_3 > 0$, then $x \in \mathcal{B}_r(E)$ by continuity of the functions u and v . This property is also immediately verified when $x_3 = 0$. As a result, if we let $a_n \uparrow a$, then we see that $\mathcal{B}_r(E)$ is convex due to the convexity of each of the sets $\mathcal{B}_r(E(n))$.

To show this, assume first that $A \leq 0$. In this case, we easily see that $ABt^2 + B^2 \leq B^2 \leq (Ct^2 + Dt + B)^2$ so that (15) holds. Then, assume that $A > 0$ and note that

$$\phi'(t) = \frac{-2ACt^3 + 2B(A - 2C)t - 2BD}{(Ct^2 + Dt + B)^3}.$$

Since $\alpha < 5$ and $r > 1/\sqrt{5 - \alpha}$, the numerator is strictly decreasing in t and, hence, it is negative due to $-2BD \leq 0$. Similarly, since $r > 1/\sqrt{2}$, the denominator is strictly increasing in t and, hence, it is strictly positive due to $B > 0$. This establishes (15) also when $A > 0$. As a result, we get

$$\|\nabla f(x_1, x_2)\|_2^2 \leq \frac{4(1 + r^2)}{r^2} \frac{1}{1 + r^2} = \frac{4}{r^2}$$

whenever $u(x_1, x_3) - a \geq 0$, thus proving (13).

To prove (14), assume that t belongs to the interval $[-1/\sqrt{\alpha}, 1/\sqrt{\alpha}]$. In this case, the numerator of $\phi(t)$ is easily seen to be maximized by B if $A \leq 0$ and by $\frac{A}{\alpha} + B$ otherwise. At the same time, the denominator of $\phi(t)$ has its global minimum at $t = -\frac{D}{2C}$, which is larger or equal than $-1/\sqrt{\alpha}$ since $a^2\alpha \leq 1$ and $r > \sqrt{2}$. Hence, the denominator is minimized by $(B - D^2/4C)^2$. Now, if $A \leq 0$ we infer that

$$\begin{aligned} \|\nabla f(x_1, x_2)\|_2^2 &\leq \frac{4(1 + r^2)}{r^2} \frac{16(1 + r^2)(2r^2 - 1)^2}{((8 - 9\alpha a^2)r^4 + 4r^2 - 4)^2} \\ &= \frac{64(2r^4 + r^2 - 1)^2}{r^2((8 - 9\alpha a^2)r^4 + 4r^2 - 4)^2} \\ &\leq \frac{256}{r^2(8 - 9\alpha a^2)^2}. \end{aligned}$$

The last inequality is due to the fact that, by assumption, $8 - 9\alpha a^2 > 0$ and $r \geq 1$. On the other side, if $A > 0$, then we have

$$\begin{aligned} \|\nabla f(x_1, x_2)\|_2^2 &\leq \frac{4(1 + r^2)}{r^2} \frac{16\alpha r^2(2r^2 - 1)^2}{((8 - 9\alpha a^2)r^4 + 4r^2 - 4)^2} \\ &\leq \frac{64\alpha^2 r^2(2r^2 - 1)^2}{((8 - 9\alpha a^2)r^4 + 4r^2 - 4)^2} \\ &\leq \frac{256\alpha^2 r^2}{(1 + r^2)^2(8 - 9\alpha a^2)^2}. \end{aligned}$$

The second inequality follows from $\alpha(1 + r^2) \leq \alpha^2 r^2$, which holds since $A > 0$, and the last inequality is, as above, due to the fact that, by assumption, $8 - 9\alpha a^2 > 0$ and $r \geq 1$. This finally establishes the bound in (14).

4. The basic set. Consider the set $C \subset \mathbb{R}^2$ defined as the union of the following four quarters of ellipsoids:

$$\begin{aligned} E_1 &= \left\{ (x_1, x_2) \in \mathbb{R}^2; 4\left(x_1 - \frac{1}{2}\right)^2 + x_2^2 \leq 1, x_1 \in \left[\frac{1}{2}, 1\right], x_2 \in [0, 1] \right\}, \\ E_2 &= \left\{ (x_1, x_2) \in \mathbb{R}^2; \frac{4}{9}\left(x_1 - \frac{1}{2}\right)^2 + x_2^2 \leq 1, x_1 \in \left[-1, \frac{1}{2}\right], x_2 \in [0, 1] \right\}, \\ E_3 &= \left\{ (x_1, x_2) \in \mathbb{R}^2; 4\left(x_1 + \frac{1}{2}\right)^2 + x_2^2 \leq 1, x_1 \in \left[-1, -\frac{1}{2}\right], x_2 \in [-1, 0] \right\}, \\ E_4 &= \left\{ (x_1, x_2) \in \mathbb{R}^2; \frac{4}{9}\left(x_1 + \frac{1}{2}\right)^2 + x_2^2 \leq 1, x_1 \in \left[-\frac{1}{2}, 1\right], x_2 \in [-1, 0] \right\}. \end{aligned}$$

Then, let $r > 0$ and consider the set $\mathcal{B}_r(C)$. This set has the same structure as the set \mathcal{B}_r considered above once we note that $\mathcal{B}_r = \mathcal{B}_r(E)$ where E was a circle. The set $\mathcal{B}_r(C)$ is clearly closed but its convexity is not a priori obvious.

5. Convexity of $\mathcal{B}_r(C)$. We prove that $\mathcal{B}_r(E_1 \cup E_2)$ is convex. By symmetry, this will imply that $\mathcal{B}_r(C)$ is also convex. Note that the half ellipsoid

$$E'_1 = \left\{ (u, v) \in \mathbb{R}^2; 4 \left(u - \frac{1}{2} \right)^2 + v^2 \leq 1, u \in [-1, 1], v \in [0, 1] \right\}$$

is contained in the half ellipsoid

$$E'_2 = \left\{ (u, v) \in \mathbb{R}^2; \frac{4}{9} \left(u - \frac{1}{2} \right)^2 + v^2 \leq 1, u \in [-1, 1], v \in [0, 1] \right\}$$

so that $\mathcal{B}_r(E'_1) \subset \mathcal{B}_r(E_1 \cup E_2) \subset \mathcal{B}_r(E'_2)$. Recall that, as established above, the set $\mathcal{B}_r(E'_2)$ is convex. Take any $x, y \in \mathcal{B}_r(E_1 \cup E_2)$ and note that any convex combination of x and y belongs to $\mathcal{B}_r(E'_2)$. Since the equation

$$u(\lambda x_1 + (1 - \lambda)y_1, \lambda x_3 + (1 - \lambda)y_3) = \frac{1}{2}$$

has at most two solutions $\lambda \in [0, 1]$, the segment with extremes x and y is divided in at most three subsegments where $u(x_1, x_3) - \frac{1}{2}$ has a constant sign. Positive subsegments are contained in $\mathcal{B}_r(E'_1)$ and, hence, belong to $\mathcal{B}_r(C)$. Negative subsegments are contained in

$$\mathcal{B}_r(E'_2) \cap \left\{ x \in \mathbb{R}^3; -1 \leq u(x_1, x_3) \leq \frac{1}{2}, x_2 \geq 0 \right\},$$

which is also contained in $\mathcal{B}_r(C)$. This establishes the convexity of $\mathcal{B}_r(C)$.

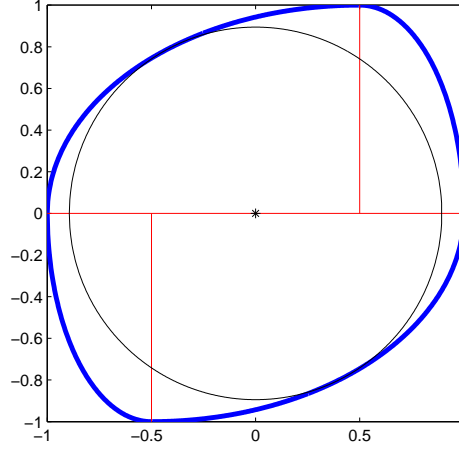


Figure 3: The set C as a union of four quarters of ellipsoids.

6. The boundary of $\mathcal{B}_r(C)$ is smooth. Let F_1 and F_2 be the functions associated to $\mathcal{B}_r(E_1)$ and $\mathcal{B}_r(E_2)$ as in (10) and note that they coincide on the set $\{x \in \mathbb{R}^3; u(x_1, x_3) = \frac{1}{2}\}$. Moreover, for every $x \in \mathbb{R}^3$ we have

$$\nabla F_1(x) = \begin{pmatrix} 4r^2 x_3 (2u(x_1, x_3) - 1) \frac{\partial u(x_1, x_3)}{\partial x_1} \\ 2x_2 \\ 4r^2 (u(x_1, x_3) - 1) u(x_1, x_3) + 4r^2 x_3 (2u(x_1, x_3) - 1) \frac{\partial u(x_1, x_3)}{\partial x_3} \end{pmatrix}$$

and

$$\nabla F_2(x) = \begin{pmatrix} \frac{4}{9} r^2 x_3 (2u(x_1, x_3) - 1) \frac{\partial u(x_1, x_3)}{\partial x_1} \\ 2x_2 \\ \frac{4}{9} r^2 (u(x_1, x_3) - 2)(u(x_1, x_3) + 1) + \frac{4}{9} r^2 x_3 (2u(x_1, x_3) - 1) \frac{\partial u(x_1, x_3)}{\partial x_3} \end{pmatrix}$$

which trivially match if $u(x_1, x_3) = 1/2$. Finally, take an arbitrary $(x_1, x_3) \in \mathbb{R}^2$ with $\frac{1}{2} \leq u(x_1, x_3) \leq 1$ and $F_1(x_1, 0, x_3) = 0$. In this case, it is easy to see that $x_1 = \sqrt{1 + r^2 x_3}$ so that, in fact, $u(x_1, x_3) = 1$. In addition, we have $u(-x_1, x_3) = -1$ as well as

$$\frac{\partial u(-x_1, x_3)}{\partial x_1} = \frac{\partial u(x_1, x_3)}{\partial x_1} \quad \text{and} \quad \frac{\partial u(-x_1, x_3)}{\partial x_3} = -\frac{\partial u(x_1, x_3)}{\partial x_3}.$$

As a result, we can use the above gradient formula to obtain

$$\begin{aligned} \frac{\partial F_2(-\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_1} &= -\frac{12}{9} r^2 x_3 \frac{\partial u(x_1, x_3)}{\partial x_1} = -\frac{1}{3} \frac{\partial F_1(\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_1}, \\ \frac{\partial F_2(-\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_2} &= 0 = \frac{\partial F_1(\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_2}, \\ \frac{\partial F_2(-\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_3} &= \frac{12}{9} r^2 x_3 \frac{\partial u(x_1, x_3)}{\partial x_3} = \frac{1}{3} \frac{\partial F_1(\sqrt{1 + r^2 x_3}, 0, x_3)}{\partial x_3}. \end{aligned}$$

This shows that the boundary of $\mathcal{B}_r(C)$ is smooth when $x_2 = 0$.

7. “Monotonicity” of $\mathcal{B}_r(C)$. For any $0 < R \leq \frac{7}{16}r$ consider the ice-cream cone

$$\mathcal{K}_R = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 \leq R^2 x_3^2\}.$$

We claim that

$$(\mathcal{B}_r(C) + \mathcal{K}_R) \cap \{x \in \mathbb{R}^3; x_3 \leq 1\} = \mathcal{B}_r(C). \quad (16)$$

To this effect, let f_3 be the function associated to $\mathcal{B}_r(E_3)$ as in (12) and note that $R \leq \frac{r}{2}$. As a result of the bound established in (13), it follows that

$$R \leq \frac{1}{\|\nabla f_3(x_1, x_2)\|_2}$$

for every $x \in \text{bd } \mathcal{B}_r(E_3)$ with $x_3 > 0$. The same bound holds, by symmetry, if we replace E_3 by E_1 . Similarly, if f_4 is the function associated to $\mathcal{B}_r(E_4)$ as in (12), then $R \leq \frac{7}{16}r$ implies that

$$R \leq \frac{1}{\|\nabla f_4(x_1, x_2)\|_2}$$

for every $x \in \text{bd } \mathcal{B}_r(E_4)$ with $x_3 > 0$ by (14). The same bound holds, by symmetry, if we replace E_4 by E_2 . Then, one can easily prove (16) in the same spirit as the proof of (6).

8. A continuous selection does not exist. We denote by Φ the isometry introduced in (8). Let $r = 16$ and consider the convex acceptance set $\mathcal{A} \subset \mathbb{R}^3$ defined by

$$\mathcal{A} = \Phi(\mathcal{B}_r(C)) + \mathbb{R}_+^3.$$

Moreover, as above, consider the space of eligible payoffs

$$\mathcal{M} = \Phi(\{x \in \mathbb{R}^3; x_2 = 0\})$$

and the pricing functional $\pi : \mathcal{M} \rightarrow \mathbb{R}$ given by

$$\pi(\Phi(x)) = \frac{x_3}{\sqrt{3}}, \quad x \in \Phi^{-1}(\mathcal{M}).$$

Set $R = \sqrt{2}$ so that $\Phi^{-1}(\mathbb{R}_+^3) \subset \mathcal{K}_R$. Since $R \leq \frac{7}{16}r$, it follows from (16) that

$$(\mathcal{B}_r(C) + \Phi^{-1}(\mathbb{R}_+^3)) \cap \{x \in \mathbb{R}^3; x_3 \leq 1\} = \mathcal{B}_r(C).$$

Now, consider the sequence $(x_n) \subset \mathbb{R}^3$ satisfying

$$x_{2n-1} = (0, -r/\sqrt{n}, 0) \quad \text{and} \quad x_{2n} = (0, r/\sqrt{n}, 0).$$

Following the same argument as in the above-mentioned example we easily obtain

$$\mathcal{E}_\rho(\Phi(x_{2n-1})) = \left\{ \Phi \left(-\frac{1}{2} \sqrt{1 + \frac{r^2}{n}}, -\frac{r}{\sqrt{n}}, \frac{1}{n} \right) \right\}$$

and similarly

$$\mathcal{E}_\rho(\Phi(x_{2n})) = \left\{ \Phi \left(\frac{1}{2} \sqrt{1 + \frac{r^2}{n}}, \frac{r}{\sqrt{n}}, \frac{1}{n} \right) \right\}$$

for every $n \in \mathbb{N}$. Since $x_n \rightarrow 0$ but we clearly have

$$\left(-\frac{1}{2} \sqrt{1 + \frac{r^2}{n}}, -\frac{r}{\sqrt{n}}, \frac{1}{n} \right) \rightarrow \left(-\frac{1}{2}, 0, 0 \right) \quad \text{and} \quad \left(\frac{1}{2} \sqrt{1 + \frac{r^2}{n}}, \frac{r}{\sqrt{n}}, \frac{1}{n} \right) \rightarrow \left(\frac{1}{2}, 0, 0 \right),$$

it follows that \mathcal{E}_ρ is neither inner semicontinuous nor can it admit any continuous selection.

References

- [1] Aliprantis, Ch.D., Border, K.C.: *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd ed., Springer, Berlin (2006)
- [2] Arai, T., Fukasawa, M.: Convex risk measures for good deal bounds, *Mathematical Finance*, 24(3), 464-484 (2014)
- [3] Armenti, Y., Crépey, S., Drapeau, S., Papapantoleon, A.: Multivariate shortfall risk allocation and systemic risk, arXiv: 1507.05351 (2016)
- [4] Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk, *Mathematical Finance*, 9, 203-228 (1999)
- [5] Artzner, Ph., Delbaen, F., Koch-Medina, P.: Risk measures and efficient use of capital, *ASTIN Bulletin*, 39 (1), 101-116 (2009)
- [6] Aubin, J.-P., Ekeland, I.: *Applied Nonlinear Analysis*, Dover Publications (2006)
- [7] Auslender, A., Teboulle, M.: *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer (2003)
- [8] Barvinok, A.: *A Course in Convexity*, Springer (2012)
- [9] Bertsekas, D.P., Nedić, A., Ozdaglar, A.E.: *Convex Analysis and Optimization*, Athena Scientific (2003)
- [10] Biagini, F., Fouque, J.-P., Frittelli, M., Meyer-Brandis, T.: A unified approach to systemic risk measures via acceptance sets, arXiv 1503.06354 (2016)
- [11] Carr, P., Geman, H., Madan, D.B.: Pricing and hedging in incomplete markets, *Journal of Financial Economics*, 62(1), 131-167 (2001)
- [12] Cochrane, J.H., Saa-Requejo, J.: Beyond arbitrage: good-deal asset price bounds in incomplete markets, *The Journal of Political Economy*, 108(1), 79-119 (2000)
- [13] Crouzeix, J.-P.: Conditions for quasiconvexity of convex functions, *Mathematics of Operations Research*, 5(1), 120-125 (1980)
- [14] Dieudonné, J.: Sur la séparation des ensembles convexes, *Mathematische Annalen*, 163(1), 1-3 (1966)
- [15] Farkas, W., Koch-Medina, P., Munari, C.: Beyond cash-additive risk measures: when changing the numéraire fails, *Finance & Stochastics*, 18, 145-173 (2014)
- [16] Farkas, W., Koch-Medina, P., Munari, C.: Measuring risk with multiple eligible assets, *Mathematics and Financial Economics*, 9(1), 3-27 (2015)
- [17] Feinstein, Z., Rudloff, B., Weber, S.: Measures of systemic risk, arXiv 1502.07961 (2016)
- [18] Föllmer, H., Schied, A.: Convex measures of risk and trading constraints, *Finance and Stochastics*, 6(4), 429-447 (2002)

- [19] Föllmer, H., Schied, A.: *Stochastic Finance: An Introduction in Discrete Time*, 3rd edition, de Gruyter (2011)
- [20] Frittelli, M., Scandolo, G.: Risk measures and capital requirements for processes, *Mathematical Finance*, 16(4), 589-612 (2006)
- [21] Madan, D.B., and Cherny, A.: Markets as a counterparty: an introduction to conic finance, *International Journal of Theoretical and Applied Finance*, 13(8), 1149-1177 (2010)
- [22] Rockafellar, R.T., Wets, R. J.-B.: *Variational Analysis*, Springer (2009)
- [23] Schrijver, A.: *The Theory of Linear and Integer Programming*, Wiley (1998)